STEP: Algebra – Ideas & Exercises (19 pages; 22/7/25)

(1) STEP questions frequently involve a large amount of algebra; especially Mechanics or Probability questions where the main idea involved may be fairly straightforward.

(2) There may be a large number of equations, but it will often be possible to employ some form of shortcut; eg concentrating on eliminating one particular variable. Some STEP questions rely on the fact that a result emerges fortuitously; ie there would normally be no guarantee that a useful result could be obtained by simply eliminating a particular variable: we are relying on an implication in the question that a useful result happens to exist. (See STEP 2023, P2, Q9(i), for example.)

(3) Potential pitfalls

(i) Beware of losing a solution of an equation by dividing out a factor.

(ii) Beware of spurious solutions (eg STEP 2011/P2/Q1(ii))

Solve the equation $x - \sqrt{x} = 6$

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Solution

Let $f(x) = x - \sqrt{x} - 6$

$f(x) = 0 \Rightarrow x - 6 = \sqrt{x}$

$\Rightarrow (x-6)^2 = x$, but this may include spurious solutions

[of $x - 6 = -\sqrt{x}$]

$\Rightarrow x^2 - 13x + 36 = 0$

$\Rightarrow (x-9)(x-4) = 0$

$\Rightarrow x = 9 \; or \; x = 4$

$f(9) = 0$ & $f(4) = -4$

Thus the only solution is $x = 9$

[Let $g(x) = x + \sqrt{x} - 6 = 0$

Then $g(x) = 0 \Rightarrow (x-6)^2 = x$ as well

$g(9) \neq 0, and \; g(4) = 0$]

Alternatively: Let $y = \sqrt{x}$, so that

$x - \sqrt{x} - 6 = 0 \Rightarrow y^2 - y - 6 = 0$

$\Rightarrow (y+2)(y-3) = 0$

$\Rightarrow y = -2 \left(\text{reject as } \sqrt{\text{x}} \text{ must be} \geq 0\right) or \; y = 3$

Solve the equation $\sqrt{2x+3} + \sqrt{x+1} = \sqrt{7x+4}$

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Solution

$\sqrt{2x+3} + \sqrt{x+1} = \sqrt{7x+4}$ (*)

$\Rightarrow (2x+3) + 2\sqrt{(2x+3)(x+1)} + (x+1) = 7x+4$

(incl. possible spurious sol'ns)

$\Rightarrow 2\sqrt{(2x+3)(x+1)} = 4x$

$\Rightarrow (2x+3)(x+1) = 4x^2$

$\Rightarrow 2x^2 - 5x - 3 = 0$

$\Rightarrow (2x+1)(x-3) = 0$

$\Rightarrow x = -\frac{1}{2}\ or\ 3$

But only $x = 3$ satisfies (*)

[$x = -\frac{1}{2}$ is a sol'n of $2\sqrt{(2x+3)(x+1)} = -4x$]

Show that $\frac{sec\theta+1-tan\theta}{sec\theta+1+tan\theta} \equiv sec\theta - tan\theta$

Solution

Equivalently, show that $\frac{sec\theta+1-tan\theta}{sec\theta+1+tan\theta}-(sec\theta-tan\theta)\equiv 0$:

LHS $=\frac{(sec\theta+1-tan\theta)-(sec\theta-tan\theta)(sec\theta+1+tan\theta)}{sec\theta+1+tan\theta}$

Numerator $=(sec\theta+1-tan\theta)$

$-(sec\theta-tan\theta)(sec\theta+tan\theta)-(sec\theta-tan\theta)$

$=(sec\theta+1-tan\theta)-(sec^2\theta-tan^2\theta)-(sec\theta-tan\theta)$

$=(sec\theta+1-tan\theta)-1-(sec\theta-tan\theta)=0$

Factorise $x^3 - y^3$ and $x^3 + y^3$ in the form

$(x - y)(\dots)$ or $(x + y)(\dots)$

Solution

Write $f(x) = x^3 - y^3$

As $f(y) = 0$ for all y, $(x - y)$ is a factor,

and $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

But $f(-y) \neq 0$ (unless $y = 0$), so that $(x + y)$ isn't a factor.

Similarly, $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$,

But $(x - y)$ isn't a factor.

Repeat for a general even number n.

Solution

$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

Or $(x + y)(x^{n-1} - x^{n-2}y + \cdots + xy^{n-2} - y^{n-1})$

[Consider $x^2 - y^2 = (x - y)(x + y)$]

And $x^n + y^n > 0$ (unless $x = y = 0$), and so neither $(x + y)$ nor $(x - y)$ are factors.

(i) Find an expansion for $(a + b + c)^3$, and give a justification for the coefficients.

(ii) Extend this to $(a + b + c)^4$

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Solution

(i) By an ordinary expansion:

$(a+b+c)^3 = ([a+b]+c)^3$

$= (a+b)^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$

$= (a^3 + 3a^2b + 3ab^2 + b^3) + (3a^2c + 3b^2c + 6abc)$

$+(3ac^2 + 3bc^2) + c^3$

$= (a^3 + b^3 + c^3) + 3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b)$

$+6abc$

Alternatively, this could have been deduced by noting that the terms fall into one of the 3 groups above.

Then there is only 1 way of creating an a^3 term from

$(a+b+c)(a+b+c)(a+b+c)$; namely by choosing the a from each of the 3 brackets.

There are 3 ways of creating an a^2b term: 3[number of ways of choosing the b]$\times$ 1[number of ways of choosing two as from the remaining 2 brackets].

Finally, there are 6 ways of creating an abc term: 3[number of ways of choosing the a]$\times$ 2[number of ways of choosing the b from the remaining 2 brackets]$\times$ 1[number of ways of choosing the c from the remaining bracket].

The final expression then follows by symmetry.

(ii) $(a+b+c)^4 = (a^4+b^4+c^4)$

$+4(a^3b+a^3c+b^3a+b^3c+c^3a+c^3b)$

$+6(a^2b^2+a^2c^2+b^2c^2)+12(a^2bc+b^2ac+c^2ab)$

For the a^2b^2 term etc, there are $\binom{4}{2} = 6$ ways of choosing the brackets from $(a+b+c)(a+b+c)(a+b+c)(a+b+c)$

to give a^2, and then just 1 way of obtaining the b^2 term.

For the a^2bc term etc, there are $\binom{4}{2} = 6$ ways of choosing the brackets for the a^2 again, multiplied by the 2 ways of choosing brackets for the b and c.

For further investigation: the 'trinomial' expansion of

$(a+b+c)^n$ can be shown to be $\sum_{\substack{i,j,k \\ (i+j+k=n)}} \binom{n}{i,j,k} a^i b^j c^k$,

where $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$

(with a further extension to the 'multinomial' expansion of

$(a_1+a_2+\cdots+a_m)^n$)

Converting from parametric to Cartesian form

(a) Make t the subject of one of the equations for $x\ or\ y$, and substitute for t in the other equation.

(b) Combine the equations for x & y in some way, so as to make t the subject; then substitute for t in of one of the original equations.

(c) Make $f(t)$ the subject of both of the equations for x & y, and equate the two expressions, leaving a single t in the resulting equation; then substitute for t in of one of the original equations.

Convert the following equations to Cartesian form:

$x = 2t + t^2,\ y = 2t^2 + t^3$

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Solution

$x = 2t + t^2,\ y = 2t^2 + t^3 \Rightarrow x = t(2+t), y = t^2(2+t)$

So $\frac{y}{x} = t$; then $x = 2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2$ and hence $x^3 = 2xy + y^2$

Convert the following equations to Cartesian form:

$x = 5t^2 - 4,\ y = 9t - t^3$

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Solution

$x = 5t^2 - 4,\ \ y = 9t - t^3 = t(9 - t^2)$; then $t^2 = \frac{x+4}{5}$ and also

$\frac{y}{t} - 9 = -t^2$; so $\frac{x+4}{5} = 9 - \frac{y}{t}$ and hence $\frac{y}{t} = 9 - \frac{x+4}{5} = \frac{45-x-4}{5} = \frac{41-x}{5}$, so that $t = \frac{5y}{41-x}$; then , substituting back into $x = 5t^2 - 4$, we have $x = 5\left(\frac{5y}{41-x}\right)^2 - 4$, and hence $(x+4)(41-x)^2 = 125y^2$

Given that $\frac{A}{4C-B} = B - C$ and $4C^2 = A + (4C - B)^2$,

show that $A = \frac{5}{16}B^2$

Solution

[It isn't feasible to eliminate C directly, as both equations are quadratics in C. But A can easily be made the subject of each equation, in order to find a relation between B & C, and this may enable C to be expressed in terms of B, so that C can be eliminated from one of the original equations. (The existence of the 'show that' result implies that there must be some substitution for C that will work.)

Making A the subject of the two equations, and equating the resulting expressions gives

$(4C - B)(B - C) = 4C^2 - (4C - B)^2$, so that

$4CB - 4C^2 - B^2 + BC = 4C^2 - 16C^2 - B^2 + 8CB$,

which simplifies to $8C = 3B$

[The convenient cancellation of $-B^2$ arises from the fact that the question has been engineered to work.]

Then, to eliminate C from $\frac{A}{4C-B} = B - C$:

$\frac{16A}{8C-2B} = 8B - 8C$;

$\frac{16A}{3B-2B} = 8B - 3B$;

$\frac{16A}{B} = 5B$;

$A = \frac{5}{16}B^2$, as required.