STEP: Algebra – Further Exercises (6 pages; 21/12/24)

Express $\frac{1}{(1-x^2)^2}$ in terms of partial fractions

Express $\frac{1}{(1-x^2)^2}$ in terms of partial fractions

Solution

$\frac{1}{(1-x^2)^2} = \frac{1}{(1-x)^2(1+x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x} + \frac{D}{(1+x)^2}$

so that $1 = A(1-x)(1+x)^2 + B(1+x)^2 + C(1+x)(1-x)^2 + D(1-x)^2$

Then $x = 1 \Rightarrow 1 = 4B \Rightarrow B = \frac{1}{4}$

$x = -1 \Rightarrow 1 = 4D \Rightarrow D = \frac{1}{4}$

$x = 0 \Rightarrow 1 = A + B + C + D \Rightarrow A + C = \frac{1}{2}$

Equating coefficients of $x^3 \Rightarrow 0 = -A + C$

Hence $A = C = \frac{1}{4}$ and $\frac{1}{(1-x^2)^2} = \frac{1}{4(1-x)} + \frac{1}{4(1-x)^2} + \frac{1}{4(1+x)} + \frac{1}{4(1+x)^2}$

Given that $\frac{bc-a}{1-c} = 7$ & $\frac{b^2c-a^2}{1-c} = 51$,

show that $\frac{a+7}{a^2+51} = \frac{b+7}{b^2+51}$

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Solution

$\frac{bc-a}{1-c}=7 \Rightarrow bc-a=7-7c \Rightarrow c(b+7)=7+a$

$\Rightarrow c=\frac{a+7}{b+7}$

and replacing a, b & 7 with a^2, b^2 & 51 gives $c=\frac{a^2+51}{b^2+51}$

so that $\frac{a+7}{b+7}=\frac{a^2+51}{b^2+51}$ and hence $\frac{a+7}{a^2+51}=\frac{b+7}{b^2+51}$ (since a^2+51 & b^2+51 are both non-zero)

If $\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$, $\phi = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}}$ and $w = \frac{u+v}{1+\frac{uv}{c^2}}$,

show that $\left(1 + \frac{uv}{c^2}\right)\gamma\phi = \frac{1}{\sqrt{1-\left(\frac{w}{c}\right)^2}}$

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Solution

The required result is equivalent to

$$\left(1 + \frac{uv}{c^2}\right)^2\left(1 - \left(\frac{w}{c}\right)^2\right) = \left(1 - \left(\frac{u}{c}\right)^2\right)\left(1 - \left(\frac{v}{c}\right)^2\right)$$

or $\left(1 + \frac{uv}{c^2}\right)^2\left(1 - \left(\frac{w}{c}\right)^2\right) - \left(1 - \left(\frac{u}{c}\right)^2\right)\left(1 - \left(\frac{v}{c}\right)^2\right) = 0$

$$LHS = \left\{1 + \frac{2uv}{c^2} + \frac{(uv)^2}{c^4}\right\} - \frac{(u+v)^2}{c^2} - \{1 - \frac{u^2}{c^2} - \frac{v^2}{c^2} + \frac{(uv)^2}{c^4}\}$$

$= 0$, as required.

[This is a result from Special Relativity: if spaceship C is seen by spaceship B to be moving away from it at speed v, and spaceship B is seen by spaceship A to be moving away from it (in the same direction as previously) at speed u, then Newtonian Physics gives the speed of C relative to A as just $u + v$, but according to Special Relativity it is w.

$\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$ is the Lorentz factor associated with changes in measurements of time and length for an object moving at relative speed v.]