STEP: Proof – Further Exercises (11 pages; 18/5/25)

Prove that $E' \Rightarrow L'$ is equivalent to $L \Rightarrow E$

Prove that $E' \Rightarrow L'$ is equivalent to $L \Rightarrow E$

Solution

$B \Rightarrow A$ means that $B \subseteq A$

$A' \Rightarrow B'$ means that if an event is outside A,

then it will be outside B, which also means that $B \subseteq A$

Also: If B is true, suppose that A is not true. Then, as $A' \Rightarrow B'$, there is a contradiction, as B is true. So A must be true, and hence $B \Rightarrow A$. This means that $B \Rightarrow A$ follows from the fact that $A' \Rightarrow B'$ (although this isn't the same thing as saying that the statements are equivalent).

[$A' \Rightarrow B'$ is known as a 'proof (that $B \Rightarrow A$) by contrapositive']

Suppose that a half price offer applies at selected stores of a supermarket for customers with loyalty cards.

H is “Half price offer applies”

S is “Customer shops at a selected store”

L is “Customer has a loyalty card”

Place the following statements into equivalent groups. Which ones are true?

$H \Rightarrow S$

$H \Leftarrow S$

“H is a necessary condition for S"

“S is a necessary condition for H"

“H is a sufficient condition for S"

“S is a sufficient condition for H"

“H is only true if S is true”

“S is only true if H is true”

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Solution

$H \Rightarrow S$ (true)

“S is a necessary condition for H"

“H is a sufficient condition for S"

“H is only true if S is true”

$H \Leftarrow S$ (false)

“H is a necessary condition for S"

“S is a sufficient condition for H"

“S is only

Let A be "$x = 3$", and let B be "$x^2 = 9$ "

Which of the following statements are true?

A is a necessary but not sufficient condition for B

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B is a necessary but not sufficient condition for A

B is a sufficient but not necessary condition for A

A (is true) only if B (is true)

B (is true) only if A (is true)

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A (is true) only if B (is true)

B (is true) only if A (is true)

Solution

A is a necessary but not sufficient condition for B **[false]**

A is a sufficient but not necessary condition for B **[true]**

B is a necessary but not sufficient condition for A **[true]**

B is a sufficient but not necessary condition for A **[false]**

A (is true) only if B (is true) **[true]**

B (is true) only if A (is true) **[false]**

For the following statements, group together the ones that are equivalent.

A: $X \Rightarrow Y$

B: Y is a sufficient condition for X

C: X is a necessary condition for Y

D: X is true only if Y is true

E: Y is true if X is true

F: If Y isn't true then X isn't true

G: If Y is true, then X is true

For the following statements, group together the ones that are equivalent.

A: $X \Rightarrow Y$

B: Y is a sufficient condition for X

C: X is a necessary condition for Y

D: X is true only if Y is true

E: Y is true if X is true

F: If Y isn't true then X isn't true

G: If Y is true, then X is true

Solution

A, D, E, F

B, C, G

(i) If $y = e^x sinx$, show that $\frac{dy}{dx} = \sqrt{2}\, e^x \sin\,(x + \frac{\pi}{4})$

(ii) Prove by induction that $\frac{d^n y}{dx^n} = \left(\sqrt{2}\right)^n e^x \sin\,(x + \frac{n\pi}{4})$

(i) If $y = e^x sinx$, show that $\frac{dy}{dx} = \sqrt{2}\, e^x \sin\,(x + \frac{\pi}{4})$

(ii) Prove by induction that $\frac{d^n y}{dx^n} = \left(\sqrt{2}\right)^n e^x \sin\,(x + \frac{n\pi}{4})$

Solution

(i) $\frac{dy}{dx} = e^x sinx + e^x cosx = \sqrt{2}\, e^x \{sinx\left(\frac{1}{\sqrt{2}}\right) + cosx\left(\frac{1}{\sqrt{2}}\right)\}$

$= \sqrt{2}\, e^x \{sinxcos\left(\frac{\pi}{4}\right) + cosxsin\left(\frac{\pi}{4}\right)\}$

$= \sqrt{2}\, e^x \sin\,(x + \frac{\pi}{4})$

(ii) [Show that the result is true for $n = 1$]

Now assume that the result is true for $n = k$,

so that $\frac{d^k y}{dx^k} = \left(\sqrt{2}\right)^k e^x \sin\,(x + \frac{k\pi}{4})$

Then $\frac{d^{k+1} y}{dx^{k+1}} = \left(\sqrt{2}\right)^k e^x \sin\left(x + \frac{k\pi}{4}\right) + \left(\sqrt{2}\right)^k e^x \cos\,(x + \frac{k\pi}{4})$

$= \left(\sqrt{2}\right)^{k+1} e^x \{\sin\left(x + \frac{k\pi}{4}\right)\left(\frac{1}{\sqrt{2}}\right) + \cos\,(x + \frac{k\pi}{4})\left(\frac{1}{\sqrt{2}}\right)\}$

$= \left(\sqrt{2}\right)^{k+1} e^x \{\sin\left(x + \frac{k\pi}{4}\right) cos\left(\frac{\pi}{4}\right) + \cos\,(x + \frac{k\pi}{4}) sin\left(\frac{\pi}{4}\right)\}$

$= \left(\sqrt{2}\right)^{k+1} e^x \sin\,(\left[x + \frac{k\pi}{4}\right] + \frac{\pi}{4})$

$= \left(\sqrt{2}\right)^{k+1} e^x \sin\,(x + \frac{(k+1)\pi}{4})$

[Standard wording]