STEP: Proof – Exercises (12 pages; 3/7/25)

Comment on the following:

$\sin(2x) = sin^2x$

$\Rightarrow 2sinxcosx = sin^2x$

$\Rightarrow 2cosx = sinx$

$\Rightarrow 2 = tanx$

$\Rightarrow x = tan^{-1}2$

$\sin(2x) = sin^2x$

$\Rightarrow 2sinxcosx = sin^2x$

$\Rightarrow 2cosx = sinx$ [or $sinx = 0$]

$\Rightarrow 2 = tanx$ [but consider $cosx = 0$]

$\Rightarrow x = tan^{-1}2$ [$+k\pi;\ -\frac{\pi}{2} < tan^{-1}\theta < \frac{\pi}{2}$]

Prove that, for $a, b, c > 0$, $\frac{a}{b} < \frac{a+c}{b+c} \Leftrightarrow a < b$

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Solution

$\frac{a}{b} < \frac{a+c}{b+c} \Leftrightarrow a(b+c) < b(a+c)$

$\Leftrightarrow ac < bc \Leftrightarrow a < b$

Show that if $X > 1$ & $Y > 1$, then $X + Y < XY + 1$

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Solution

$X + Y < XY + 1 \Leftrightarrow X + Y - XY - 1 < 0$

$\Leftrightarrow X(1 - Y) + Y - 1 < 0$

$\Leftrightarrow (X - 1)(1 - Y) < 0$

$\Leftrightarrow (X - 1)(Y - 1) > 0$

Then $X > 1$ & $Y > 1 \Rightarrow (X - 1)(Y - 1) > 0 \Rightarrow X + Y < XY + 1$

Given that n is a positive integer, prove that n is odd if and only if n^2 is odd.

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Solution

Part 1: To prove that n is odd $\Rightarrow n^2$ is odd

If n is odd, then it can be written as $2m + 1$, for some integer m.

Then $n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$,

so that n^2 is odd.

Thus n is odd $\Rightarrow n^2$ is odd; ie n is odd only if n^2 is odd.

Part 2: To prove that n^2 is odd $\Rightarrow n$ is odd

[Proof by contradiction]

If n^2 is odd, suppose that n is even. Then $n = 2m$, for some integer m.

But then $n^2 = (2m)^2 = 4m^2$, which is divisible by 2, and so even.

This contradicts the fact that n^2 is odd, and so n must be odd.

Thus n^2 is odd $\Rightarrow n$ is odd; ie n is odd if n^2 is odd.

[Alternatively, prove that "n not odd $\Rightarrow n^2$ is not odd"]

So n is odd if and only if n^2 is odd.

Prove that there are no positive integers m and n such that

$m^2 = n^2 + 1$

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Solution

[Proof by contradiction]

Suppose that $m^2 = n^2 + 1$, where m and n are positive integers.

Then $m^2 - n^2 = 1$,

and hence $(m - n)(m + n) = 1$

As m and n are integers, $m - n$ and $m + n$ will also be integers, and so they are either both 1 or both -1

But $m + n > 0$, so that $m - n = 1$ and $m + n = 1$

Subtracting the 1st eq'n from the 2nd gives $2n = 0$, so that $n = 0$, which contradicts the assumption that n is a positive integer.

So there are no positive integers m and n such that $m^2 = n^2 + 1$

If $x > 1$, show that $x - \sqrt{x^2 - 1} < 1$

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Solution

Suppose that $x - \sqrt{x^2 - 1} \geq 1$ (*)

Then $x - 1 \geq \sqrt{x^2 - 1}$

and so, as the RHS is non-negative, $(x - 1)^2 \geq x^2 - 1$

$\Rightarrow -2x + 1 \geq -1$

$\Rightarrow 2 \geq 2x$

$\Rightarrow x \leq 1$, which contradicts the fact that $x > 1$.

Thus (*) is not possible, and so $x - \sqrt{x^2 - 1} < 1$.