Sixth Term Examination Paper [STEP]

Mathematics 3 [9475]

2021

Examiner's Report

Mark Scheme

STEP MATHEMATICS 3

2021

Examiner's Report

STEP 3 2021

Introduction

The total entry was a marginal increase from that of 2019, that of 2020 having been artificially reduced. Comfortably more than 90% attempted one of the questions, four others were very popular, and a sixth was attempted by 70%. Every question was attempted by at least 10% of the candidature.

85% of candidates attempted no more than 7 questions, though very nearly all the candidates made genuine attempts on at most six questions (the extra attempts being at times no more than labelling a page or writing only the first line or two).

Generally, candidates should be aware that when asked to “Show that” they must provide enough working to fully substantiate their working, and that they should follow the instructions in a question, so if it says “Hence”, they should be using the previous work in the question in order to complete the next part. Likewise, candidates should be careful when dividing or multiplying, that things are positive, or at other times non-zero.

Question 1

This was the most popular question by a fair margin, being attempted by 93%, and equally was comfortably the most successful with a mean mark of slightly over 15/20. Generally, most found the equation of the normal in part (i) correctly, though the more successful candidates simplified their answer sensibly at this point and similarly with other results in the question. A number of candidates forgot the negative sign when obtaining a perpendicular gradient and merely attempted to use the reciprocal. Most used implicit differentiation in order to arrive at an expression for the gradient of the tangent to the second curve in part (i), though parametric differentiation was probably simpler. There was an equal split between those that obtained the equation of the tangent to the second curve and demonstrated that it was the same as that for the normal to the first curve, and those that demonstrated that the point given parametrically was on the normal and that the gradient of the normal and the tangent were the same.
In part (ii), surprisingly, some candidates made errors with the initial differentiation. Those that simplified their equation of the normal profited from the easier working, whichever way they then tried to obtain the perpendicular distance. About three quarters of the candidates found this distance by first finding the intersection of the normal with a perpendicular line through the origin. However, using the formula for the perpendicular distance of a point from a line was simpler. A range of other methods for this distance were seen; briefly, these were (a) simple trigonometry having sketched the normal, the axes and line's intercepts, (b) expressing the normal equation as the scalar product of vectors, (c) minimising by differentiation, or completing the square, of the distance of a general point on the normal from the origin or (d) by equating two expressions for the area of the triangle formed by the normal and the two axes. Errors in this part arose from unsimplified working complicating the issue (as already mentioned), overlooking the modulus sign in the distance formula, or calculating the distance from the origin to a point on the curve. The final requirement for the equation of a curve to which the normal found is a tangent was either not spotted by some candidates who had otherwise answered the question perfectly, or the requirement was overlooked.

Question 2

This was the fourth most popular question being attempted by very nearly four fifths of the candidates. It was the third most successful with a mean mark of just over 9/20, though very few achieved full marks. With four "Show that"s, marks were frequently lost for lack of proper justification, and with inequalities to demonstrate involving fractional quantities, positivity was often not considered, let alone proved, or stated as relevant. Even if candidates stated det(M)=0, which they sometimes didn't, only a minority of candidates realised that they had to justify using det(M)=0, and of these only some could do so convincingly; there were a number of incorrect arguments used. Some candidates sacrificed marks by, for example, attempting to show that $\frac{a^2}{(b-c)^2}+\frac{b^2}{(c-a)^2}+\frac{c^2}{(a-b)^2} \geq 2$ via a purely algebraic approach rather than using the result just found (i.e. ignoring the "hence").

For the very last part of the question, candidates used a variety of methods in order to explain why $x+y+z>2$. The most common method was to express the sum in terms of *a*, *b* and *c* and then show that this was greater than 2, but approaches using the AM-GM inequality or by splitting into different cases were sometimes used successfully.

Question 3

Whilst this was the second most popular question, being attempted by 84%, it was the fifth most successful with a mean mark a little below 8/20. Most candidates scored full marks for successfully obtaining the first result of part (i), and many gained nearly full credit for obtaining the second result of that part. As is nearly always true, the rule of thumb that it is usually easier to prove that something is greater than (or less than) zero applied here, and so those that considered $\frac{1}{2}\,(I_{n+1} + I_{n-1}) - I_n$ (and a similar expression for part (ii)) generally fared better. A small, but not insignificant number of candidates solved part (ii) by a direct method and were generally successful if they did so. Common errors when considering inequalities were failure to fully justify positivity of integrals in both parts, incorrect flows of logic, obtaining weak rather than strict inequalities, and stating inequalities that were inconsistent with the claimed ranges of validity. Otherwise, use of induction or integration by parts caused difficulties, and a number expected, when replicating the first part of working in (ii) from part (i), that there would again be an equation, and overlooked the extra term that arose in (ii). Some did not understand that $sec\,x\,cos\,\beta \leq 1$ in (ii).

Question 4

Comfortably the least popular Pure question on the paper, it was attempted by just very slightly more than a third of the candidates, which made it almost exactly the same popularity as the most popular Probability and Statistics question. With a mean score of less than 7/20, it was seventh most successful. Those candidates who engaged with the given definition of projection and followed the structure of the question generally did correct calculations of dot products and recognised the relevance of their calculations. Several candidates assumed properties of a projection, not realising that the purpose of this question was to prove properties of a projection given only a single definition. Many of these implicitly made the assumptions when drawing geometric diagrams and arguing geometrically.

Question 5

A handful of candidates more attempted this question than question 2, but with marginally less success than question 4. Nearly every candidate obtained the very first result and many then obtained $a = \frac{1}{2}$ from considering the discriminant. Finding the other values of a (1 and 5) caused many candidates difficulty which could have been overcome had they considered equating expressions for $\frac{dr}{d\theta}$. In the diagrams, the curve representing the second equation was often drawn as an ellipse, or with cusps rather than smooth indentations. On the other hand, touching points were usually well drawn. It seemed that many appreciated that the curves had symmetry but seldom referred to this in their justification. Similarly, many might have earned credit, but didn't, for indicating values of r for important points such as where the curves met the initial line or the line perpendicular to it. Few candidates found the angles of the cusp in the first two cases (especially with struggling to deal with $\arccos\left(-\frac{1}{4}\right)$, as opposed to $\arccos\left(-\frac{1}{2}\right)$).

Question 6

The seventh most popular question, it was attempted by almost 70% of candidates. However, it was fourth most successful with a mean just short of 8/20. Most candidates successfully differentiated f_α correctly to obtain the required result. Many then sketched a shifted arctan graph but frequently failed to appreciate that there were two branches to the curve with a discontinuity at $x = \tan\alpha$, and also often forgot that the range of the function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In addition, few candidates labelled all the requisite values of intercepts, the discontinuity, the asymptote, and the range on the axes. Few consequently sketched $f_\alpha(x) - f_\beta(x)$ correctly. In part (ii), many candidates incorrectly manipulated the negative sign when differentiating g, which then meant that although they sketched the section of the graph for $\frac{\pi}{2} < x < \frac{3\pi}{2}$, they did not wonder why the negative sign arose and hence failed to sketch the two constant segments of the function.

Question 7

This was the least successfully attempted Pure question with a mean score under 6/20. It was less than 4% more popular than question 4. The first part of this question was generally well attempted, with a significant number of candidates being able to correctly verify the algebraic identity utilising a number of different approaches. There were some very neat solutions, but candidates who multiplied throughout by the complex conjugate and managed to keep track of the ensuing algebra were also often successful. Candidates must make sure that when they are trying to show a given result that they fully justify their solution – in this case some candidates missed out several steps of working and so did not gain full credit. Many candidates recognised that the form of *z* meant that the number was purely imaginary, but only a few candidates gained full credit for this part of the question with many omitting the modulus signs on the cot term for the modulus, or omitting the second possible angle. Some candidates were confused by the angles present in the given form of *z* and gave the argument as $\frac{1}{2}(\phi - \theta)$. In part (ii), the approach using the result from part (i) often did not score full marks due to the fact candidates would divide by quantities without explaining why they were non-zero. Some attempted this question with vector methods without clearly setting up that they were treating a,b as vectors rather than complex numbers. They were often unclear as to whether they were actually considering vectors, or considering complex numbers, which was particularly apparent in attempts to take the dot product of vectors without including the “dot” symbol. A number of candidates attempted to work out the gradients of the two line segments and show they multiplied to give -1: unfortunately, none recognised that a number of special cases were not taken care of with this method (cases where the lines were horizontal and vertical) and so did not score highly. Some candidates took a geometrical approach which needed to be fully explained to be convincing. For part (iii), many were more successful than for (ii): they recognised that part (ii) could be applied to give the result, and those who did generally gained full, or nearly full, credit. Vector approaches and considering the gradients of the line segments were used again in this part, with some candidates repeating the work they had done in the previous part, with the same pitfalls. Many omitted the case “if b+c=0 then h=a”. Part (iv) was not attempted by a significant proportion. Of those who did attempt it, a significant number gained full credit. The most common mistake for this part of the question was candidates giving the transformation as “reflection through a point”, which did not gain them credit as this is not considered to be a “Single transformation” as requested (each point is reflected through a different line). Another common mistake was the miscalculation of the midpoint of AQ as (b+c+d-a)/2 or as (a+b+c+d)/4.

Question 8

Fifth most popular (77%), this was fourth least successful with a mean mark of six and a half. There were very few perfect attempts and a sizeable number of attempts failed to get any marks. Induction in both parts (i) and (iii) was generally executed very well, however marks were frequently lost for logical imprecision. A very common cause of lost marks was a lack of care with inequalities involving potentially negative numbers. In part (i), almost no candidates noticed that squaring the inequality required noting the non-negativity of the lower bound. Many candidates also had trouble with the base case, some because they were mistakenly thinking $4^0 = 0$. In part (ii), many candidates lost marks when attempting to show that the sequence $|x_n|$ remains bounded in the case $|a| < 2$, by not excluding the possibility that x_2 goes below -2 and hence diverges to positive infinity. Another common error in part (ii) was failing to make the link to the inequality in part (i). Many candidates tried to show divergence to infinity by showing that the sequence was increasing. In part (iii) most candidates worked back from the required result to find a suitable value for a. The inductive calculation was generally performed well, however plenty of candidates failed to show that their value of a worked and was greater than 2. When solving equations, it should either be checked that all the steps are reversible (in this case they were not because of a possible division by zero) or that the claimed solution does in fact work. Most attempts at the final section on convergence were informal but successful.

Question 9

Just over a fifth attempted this but it had the dubious distinction of being the least successful question with a mean score a little over 4/20. There were a number of alternative methods used for the first result, and those that were successful usually applied the sine rule or dropped perpendiculars. However, some candidates drew a triangle with angles found and wrote down sine or cosine rules with no indication of how they were to be combined thus earning very little credit. Candidates who understood the concept of restitution were usually able to complete the second part of the question without any problems. Many candidates failed on the last part of the question by trying to give verbose intuition-based arguments instead of finding a third restitution equation.

Question 10

Whilst this was the least popular question, being attempted by a tenth of the candidature, it was the sixth most successful with a mean over 7/20. Part (i) was successfully attempted by many candidates, by correctly finding the coordinates of the particle and then using differentiation and Pythagoras to find the speed as required. In part (ii), many understood that they could use conservation of energy even though they failed to justify it. Many used the appropriate circular motion formula in part (iii), but then stumbled as they lacked justification of the evaluation of their constant of integration, or the choice of sign when taking the square root. Quite a few struggled to find the link between $b - a\theta$ and the given answer, and some attempted to jump to the given answer!

Question 11

Comfortably the most popular applied question on the paper attracting slightly more than a third of candidates, it was the second most successful on the whole paper with a mean of 11/20. The quality of attempts for this question was high, with many candidates scoring full or close to full marks. Almost all candidates attempting it dealt with part (i) successfully. However, in part (ii) candidates often made incorrect conditioning arguments. The most common errors were computing $P(Z < z|Y = n)$ rather than $P(Z < z)$ and confusing $P(Z < z|Y = n)$ with $P(Z < z \text{ } and \text{ } Y = n)$. In part (iii), most candidates suitably obtained a probability density function for Z, but there were several computational mistakes in the integration by parts to evaluate the expectation. The independence argument in part (iv) was largely well executed, even when candidates had been unsuccessful in answering parts (ii) and (iii) of the question.

Question 12

A sixth of candidates attempted this, making it the second least popular, and it was the third least successful with a mean just shy of six and a half. Very few candidates obtained full marks for the very first part of the question, showing that X_{12} and X_{23} are independent; the most common error being to check only that $P(X_{12} = 1, X_{23} = 1) = P(X_{12} = 1)P(X_{23} = 1)$, rather than all four possible cases for the different values of the two random variables. However, in general, candidates engaged well with the combinatorial aspect of this part and provided sound methods for counting pairs of indices in order to obtain the mean and variance, though many did not use the fact that for independent random variables, $Var(\sum_i X_i) = \sum_i Var(X_i)$. However, part (ii) was consistently well executed, with most candidates that attempted it being successful. In part (iii), establishing non-independence was well executed, and again, as in part (i), candidates provided sound methods for counting pairs of indices.

STEP MATHEMATICS 3

2021

Mark Scheme

1. (i) $x = -4\cos^3 t$ so $\frac{dx}{dt} = 12\cos^2 t\ \sin t$ **M1**

$y = 12\sin t - 4\ \sin^3 t$ so $\frac{dy}{dt} = 12\cos t - 12\sin^2 t\ \cos t = 12\cos t\ (1-\sin^2 t) =\ 12\cos^3 t$

M1

So $\frac{dy}{dx} = \frac{12\cos^3 t}{12\cos^2 t\ \sin t} = \cot t$ **A1**

Thus the equation of the normal at $(-4\cos^3\varphi\ ,12\sin\varphi - 4\ \sin^3\varphi\ )$ is

$$y - (12\sin\varphi - 4\ \sin^3\varphi) = -\frac{1}{\cot\varphi}(x - -4\cos^3\varphi)$$

M1 A1ft

This simplifies to $x\sin\varphi + y\cos\varphi = 12\sin\varphi\cos\varphi - 4\sin^3\varphi\cos\varphi - 4\sin\varphi\cos^3\varphi$

That is $x\sin\varphi + y\cos\varphi = 8\sin\varphi\cos\varphi$ **A1 (6)**

Alternative simplification $x\tan\varphi + y = 8\sin\varphi$

For $x = 8\cos^3 t$, $\frac{dx}{dt} = -24\cos^2 t\ \sin t$ and for $y = 8\ \sin^3 t$, $\frac{dy}{dt} = 24\sin^2 t\ \cos t$

So $\frac{dy}{dx} = \frac{24\sin^2 t\ \cos t}{-24\cos^2 t\ \sin t} = -\tan t$ **M1 A1ft**

Thus the equation of the tangent to $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ at $(8\cos^3\varphi\ ,8\ \sin^3\varphi\ )$ is

$$y - 8\ \sin^3\varphi = -\tan\varphi\ (x - 8\cos^3\varphi)$$

M1

This simplifies to

$$x\sin\varphi + y\cos\varphi = 8\ \sin^3\varphi\cos\varphi +\ 8\sin\varphi\cos^3\varphi = 8\sin\varphi\cos\varphi\ (\sin^2\varphi + \cos^2\varphi)$$

That is $x\sin\varphi + y\cos\varphi = 8\sin\varphi\cos\varphi$ as required. **A1 (4)**

Alternative 1

the normal is a tangent to the second curve if it has the same gradient and the point $(8\cos^3\varphi\ ,8\ \sin^3\varphi\ )$ lies on the normal. **M1**

Gradient working as before **M1A1ft**

Substitution $x\sin\varphi + y\cos\varphi = 8\sin\varphi\ \cos^3\varphi + 8\ \sin^3\varphi\ \cos\varphi = 8\sin\varphi\cos\varphi\,(\sin^2\varphi + \cos^2\varphi) = 8\sin\varphi\cos\varphi$ as required or $x\tan\varphi + y = 8\sin\varphi\cos\varphi\,(\sin^2\varphi + \cos^2\varphi)$ **A1**

Alternative 2

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0$$

M1

(ii) $x = \cos t + t\sin t$ so $\frac{dx}{dt} = -\sin t + t\cos t + \sin t = t\cos t$

$y = \sin t - t\cos t$ so $\frac{dy}{dt} = \cos t - \cos t + t\sin t = t\sin t$ **M1**

So $\frac{dy}{dx} = \tan t$ **A1**

Thus the equation of the normal at $(\cos\varphi + \varphi\sin\varphi\ ,\ \sin\varphi - \varphi\cos\varphi)$ is

$$y - (\sin\varphi - \varphi\cos\varphi) = -\cot\varphi\big(x - (\cos\varphi + \varphi\sin\varphi)\big)$$

M1 A1ft

This simplifies to $x\cos\varphi + y\sin\varphi = 1$ **A1 (5)**

Alternatives which can be followed through to perpendicular distance step, or alternative method # are

$x + y\tan\varphi = \sec\varphi$ and $x\cot\varphi + y = \csc\varphi$

The distance of $(0,0)$ from $x\cos\varphi + y\sin\varphi = 1$ is $\left|\frac{-1}{\sqrt{\cos^2\varphi+\sin^2\varphi}}\right| = 1$

M1 A1ft **A1**

Alternatively, the perpendicular to $x\cos\varphi + y\sin\varphi = 1$ through $(0,0)$ is

$y\cos\varphi - x\sin\varphi = 0$, and these two lines meet at $(\cos\varphi\ ,\ \sin\varphi)$

M1 A1ft

which is a distance $\sqrt{\cos^2\varphi + \sin^2\varphi} = 1$ from $(0,0)$. **A1**

So the curve to which this normal is a tangent is a circle centre $(0,0)$, radius 1 which is thus $x^2 + y^2 = 1$ **M1 A1 (5)**

2. (i) $\begin{pmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a-(b-c)x \\ b-(c-a)y \\ c-(a-b)z \end{pmatrix} = \begin{pmatrix} a-a \\ b-b \\ c-c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ as required. **M1 A1***

As a, b and c are distinct, they cannot all be zero. If M^{-1} exists $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = M^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ which is a contradiction.

So, M^{-1} does not exist and thus $det\begin{pmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{pmatrix} = 0$, **M1**

i.e. $1 - xyz + xyz + yz + zx + xy = 0$, (Sarus)

or $1(1+yz) - -x(y-yz) + x(yz+z) = 0$ (by co-factors) **M1**

which simplifies to

$yz + zx + xy = -1$ **A1 * (5)**

$(x+y+z)^2 \geq 0$

So $x^2 + y^2 + z^2 + 2yz + 2zx + 2xy \geq 0$ **M1**

and so $x^2 + y^2 + z^2 \geq 2$ **A1* (2)**

(ii) $\begin{pmatrix} 2 & -x & -x \\ -y & 2 & -y \\ -z & -z & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a-(b+c)x \\ 2b-(c+a)y \\ 2c-(a+b)z \end{pmatrix} = \begin{pmatrix} 2a-2a \\ 2b-2b \\ 2c-2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

B1 **M1** **A1**

As a, b and c are positive, they cannot all be zero. Thus as $\begin{pmatrix} 2 & -x & -x \\ -y & 2 & -y \\ -z & -z & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

as in part (i), $det\begin{pmatrix} 2 & -x & -x \\ -y & 2 & -y \\ -z & -z & 2 \end{pmatrix} = 0$,

i.e. $8 - xyz - xyz - 2yz - 2zx - 2xy = 0$, that is **M1 A1**

$xyz + yz + zx + xy = 4$ **A1* (6)**

$$(x+1)(y+1)(z+1) = xyz + zx + xy + x + y + z + 1 = 4 + x + y + z + 1 > 5$$

M1 **A1**

because as a, b, and c are all positive, so are x, y and z. **E1**

Thus $\left(\frac{2a}{b+c}+1\right)\left(\frac{2b}{c+a}+1\right)\left(\frac{2c}{a+b}+1\right) > 5$

Multiplying by $(b+c)(c+a)(a+b)$, all three factors of which are positive, gives

$(2a+b+c)(a+2b+c)(a+b+c) > 5(b+c)(c+a)(a+b)$ as required. **A1* (4)**

$x = \frac{2a}{b+c} > \frac{2a}{a+b+c}$ as a, b, and c are positive, and similarly both, $y > \frac{2b}{a+b+c}$ and $z > \frac{2c}{a+b+c}$

M1

Thus $4 + x + y + z + 1 > 4 + \frac{2a}{a+b+c} + \frac{2b}{a+b+c} + \frac{2c}{a+b+c} + 1 = 4 + \frac{2(a+b+c)}{a+b+c} + 1 = 7$

dM1

and thus following the argument used to obtain the previous result

$(2a+b+c)(a+2b+c)(a+b+c) > 7(b+c)(c+a)(a+b)$ as required.

A1* (3)

3. (i)

$$\frac{1}{2}(I_{n+1}+I_{n-1})=\frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n+1}+(\sec x+\tan x)^{n-1}dx$$

$$=\frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n-1}\left((\sec x+\tan x)^2+1\right)dx$$

M1

$$=\frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n-1}(\sec^2 x+2\sec x\tan x+\tan^2 x+1)\,dx$$

$$=\int_0^\beta (\sec x+\tan x)^{n-1}(\sec^2 x+\sec x\tan x)\,dx$$

M1

$$=\left[\frac{1}{n}(\sec x+\tan x)^n\right]_0^\beta=\frac{1}{n}\left((\sec\beta+\tan\beta)^n-1\right)$$

M1 A1 ***A1 (5)**

as required.

$$\frac{1}{2}(I_{n+1}+I_{n-1})-I_n=\frac{1}{2}(I_{n+1}-2I_n+I_{n-1})$$

$$=\frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n+1}-2(\sec x+\tan x)^n+(\sec x+\tan x)^{n-1}dx$$

M1

$$=\frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n-1}\left((\sec x+\tan x)-1\right)^2dx$$

M1 A1

$\left((\sec x+\tan x)-1\right)^2>0$ for all x>0

$\sec x\geq 1$ for $0\leq x<\frac{\pi}{2}$ and hence for $0\leq \mathrm{x}<\beta$ and similarly $\tan x\geq 0$, and thus also $(\sec x+\tan x)^{n-1}>0$. **E1**

Therefore, $\frac{1}{2}(I_{n+1}+I_{n-1})-I_n>0$, **A1**

and so $I_n<\frac{1}{2}(I_{n+1}+I_{n-1})=\frac{1}{n}\left((\sec\beta+\tan\beta)^n-1\right)$ as required. **M1 *A1 (7)**

Alternative 1: it has already been shown that

$\frac{1}{2}\left(I_{n+1}+I_{n-1}\right)=\int_0^\beta(\sec x+\tan x)^{n-1}(\sec^2 x+\sec x\tan x)\,dx$

$$=\int_0^\beta \sec x\,(\sec x+\tan x)^n\,dx$$

which is greater than I_n as the expression being integrated is greater than $(\sec x+\tan x)^n$ because $\sec x>0$ over this domain.

Alternative 2:-

$$I_{n+1}-I_n=\int_0^\beta(\sec x+\tan x)^n(\sec x+\tan x-1)\,dx$$

$$I_n-I_{n-1}=\int_0^\beta(\sec x+\tan x)^{n-1}(\sec x+\tan x-1)\,dx$$

M1 A1 A1

For $0<\mathrm{x}<\beta$, $\sec x>1$, $\tan x>0$ so $\sec x+\tan x>1$ **E1** and thus $I_{n+1}-I_n>I_n-I_{n-1}$ **A1**

and so $I_n\le\frac{1}{2}\left(I_{n+1}+I_{n-1}\right)=\frac{1}{n}\left((\sec\beta+\tan\beta)^n-1\right)$ **M1 *A1 (7)**

(ii) $\frac{1}{2}\left(J_{n+1}+J_{n-1}\right)=\frac{1}{2}\int_0^\beta(\sec x\cos\beta+\tan x)^{n+1}+(\sec x\cos\beta+\tan x)^{n-1}dx$

$$=\frac{1}{2}\int_0^\beta(\sec x\cos\beta+\tan x)^{n-1}\left((\sec x\cos\beta+\tan x)^2+1\right)dx$$

M1

$$=\frac{1}{2}\int_0^\beta(\sec x\cos\beta+\tan x)^{n-1}\left(\sec^2 x\cos^2\beta+2\sec x\cos\beta\tan x+\tan^2 x+1\right)dx$$

$$=\frac{1}{2}\int_0^\beta(\sec x\cos\beta+\tan x)^{n-1}\left(\sec^2 x\,(1-\sin^2\beta)+2\sec x\cos\beta\tan x+\tan^2 x+1\right)dx$$

$$=\int_0^\beta(\sec x\cos\beta+\tan x)^{n-1}\left((\sec^2 x+\sec x\cos\beta\tan x)-\sec^2 x\sin^2\beta\right)dx$$

M1

$$\int_0^\beta(\sec x\cos\beta+\tan x)^{n-1}\left(\sec^2 x+\sec x\cos\beta\tan x\right)dx=\left[\frac{1}{n}(\sec x\cos\beta+\tan x)^n\right]_0^\beta$$

M1

$$= \frac{1}{n}\left((1+\tan\beta)^n - \cos^n\beta\right)$$

A1

$$\int_0^\beta (\sec x\cos\beta + \tan x)^{n-1}\ \sec^2 x \sin^2\beta\, dx > 0$$

by a similar argument to part (i), namely $\sec^2 x \sin^2\beta > 0$ for any x, and $\sec x\cos\beta + \tan x > 0$ as $\sec x > 0$ and $\tan x \geq 0$ for $0 \leq \mathrm{x} < \beta < \frac{\pi}{2}$ **E1**

Hence $\frac{1}{2}\,(J_{n+1} + J_{n-1}) < \frac{1}{n}\left((1+\tan\beta)^n - \cos^n\beta\right)$ **A1**

But

$$\frac{1}{2}\,(J_{n+1} + J_{n-1}) - J_n == \frac{1}{2}\int_0^\beta (\sec x\cos\beta + \tan x)^{n-1}\left((\sec x\cos\beta + \tan x) - 1\right)^2 dx > 0$$

M1

as before, and thus $J_n < \frac{1}{2}\,(J_{n+1} + J_{n-1}) < \frac{1}{n}\left((1+\tan\beta)^n - \cos^n\beta\right)$ as required. ***A1 (8)**

4. (i)

$\boldsymbol{m}.\boldsymbol{a} = \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{a} = \frac{1}{2}(1+\boldsymbol{a}.\boldsymbol{b}) = m\cos\alpha$ where α is the non-reflex angle between **a** and **m**

$\boldsymbol{m}.\boldsymbol{b} = \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{b} = \frac{1}{2}(1+\boldsymbol{a}.\boldsymbol{b}) = m\cos\beta$ where α is the non-reflex angle between $\boldsymbol{b}$ and **m**

M1 A1

Thus $\cos\alpha = \cos\beta$ and so $\alpha = \beta$ as for $0 \le \tau \le \pi$, there is only one value of τ for any given value of $\cos\tau$. **E1 (3)**

(ii) $\boldsymbol{a_1}.\boldsymbol{c} = (\boldsymbol{a}-(\boldsymbol{a}.\boldsymbol{c})\boldsymbol{c}).\boldsymbol{c} = \boldsymbol{a}.\boldsymbol{c} - \boldsymbol{a}.\boldsymbol{c}\,\boldsymbol{c}.\boldsymbol{c} = 0$ as required. ***B1**

$\boldsymbol{a}.\boldsymbol{c} = \cos\alpha$, $\boldsymbol{b}.\boldsymbol{c} = \cos\beta$, $\boldsymbol{a}.\boldsymbol{b} = \cos\theta$

$\boldsymbol{a_1} = \boldsymbol{a} - (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{c}$ and $\boldsymbol{b_1} = \boldsymbol{b} - (\boldsymbol{b}.\boldsymbol{c})\boldsymbol{c}$

$$|\boldsymbol{a_1}|^2 = \boldsymbol{a_1}.\boldsymbol{a_1} = (\boldsymbol{a}-(\boldsymbol{a}.\boldsymbol{c})\boldsymbol{c}).(\boldsymbol{a}-(\boldsymbol{a}.\boldsymbol{c})\boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{a} - 2\boldsymbol{a}.\boldsymbol{c}\,\boldsymbol{a}.\boldsymbol{c} + \boldsymbol{a}.\boldsymbol{c}\,\boldsymbol{a}.\boldsymbol{c}\,\boldsymbol{c}.\boldsymbol{c}$$

$$= 1 - 2\cos^2\alpha + \cos^2\alpha = \sin^2\alpha$$

M1

and so. as α is acute, $|\boldsymbol{a_1}| = \sin\alpha$ as required. ***A1**

$$\boldsymbol{a_1}.\boldsymbol{b_1} = (\boldsymbol{a}-(\boldsymbol{a}.\boldsymbol{c})\boldsymbol{c}).(\boldsymbol{b}-(\boldsymbol{b}.\boldsymbol{c})\boldsymbol{c}) = \boldsymbol{a}.\boldsymbol{b} - 2(\boldsymbol{a}.\boldsymbol{c})(\boldsymbol{b}.\boldsymbol{c}) + (\boldsymbol{a}.\boldsymbol{c})(\boldsymbol{b}.\boldsymbol{c})(\boldsymbol{c}.\boldsymbol{c})$$

$$= \cos\theta - \cos\alpha\cos\beta$$

M1 A1

but also, $\boldsymbol{a_1}.\boldsymbol{b_1} = \sin\alpha\sin\beta\cos\varphi$ **B1 M1**

and hence,

$$\cos\varphi = \frac{\cos\theta - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}$$

as required. ***A1 (8)**

(iii) $\boldsymbol{m_1} = \boldsymbol{m} - (\boldsymbol{m}.\boldsymbol{c})\boldsymbol{c} = \frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}) - \left(\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{c}\right)\boldsymbol{c} = \frac{1}{2}(\boldsymbol{a_1}+\boldsymbol{b_1})$ **B1**

$\boldsymbol{m_1}$ bisects the angle between $\boldsymbol{a_1}$ and $\boldsymbol{b_1}$ if and only if

$$\frac{\boldsymbol{m_1}.\boldsymbol{a_1}}{\sin\alpha} = \frac{\boldsymbol{m_1}.\boldsymbol{b_1}}{\sin\beta}$$

M1

Thus, multiplying through by $2\sin\alpha\sin\beta$,

$$(\boldsymbol{a_1}+\boldsymbol{b_1}).\boldsymbol{a_1}\sin\beta = (\boldsymbol{a_1}+\boldsymbol{b_1}).\boldsymbol{b_1}\sin\alpha$$

A1

$$(\sin^2\alpha + \boldsymbol{a_1}.\boldsymbol{b_1})\sin\beta = (\sin^2\beta + \boldsymbol{a_1}.\boldsymbol{b_1})\sin\alpha$$

M1 A1

So

$$(\boldsymbol{a_1}.\boldsymbol{b_1} - \sin\alpha\sin\beta)(\sin\alpha - \sin\beta) = 0$$

A1

and thus, $\sin\alpha = \sin\beta$ in which case $\alpha = \beta$ as both angles are acute, ***A1**

or $\cos\theta - \cos\alpha\cos\beta = \sin\alpha\sin\beta$, meaning that $\cos\theta = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

M1 *A1 (9)

5. (i) The curves meet when $a + 2\cos\theta = 2 + \cos 2\theta$

That is, $a + 2\cos\theta = 2 + 2\cos^2\theta - 1$ or as required, **B1** $2\cos^2\theta - 2\cos\theta + 1 - a = 0$

The curves touch if this quadratic has coincident roots, **M1** i.e. if $4 - 8(1-a) = 0 \Rightarrow a = \frac{1}{2}$, ***A1** or if $\cos\theta = \pm 1$, **M1** in which cases $a = 1$ **A1** or $a = 5$. **A1 (6)**

Alternatively, for the curves to touch, they must have the same gradient, so differentiating,

$$-2\sin\theta = -2\sin 2\theta = -4\sin\theta\ \cos\theta$$

M1

in which case, either $\sin\theta = 0$ giving $\cos\theta = \pm 1$, **M1** in which cases $a = 1$ **A1** or $a = 5$, **A1** or $\cos\theta = \frac{1}{2}$ in which case $a = \frac{1}{2}$. ***A1 (6)**

(ii) If $a = \frac{1}{2}$ then at points where they touch, $\cos\theta = \frac{1}{2}$ so $\theta = \pm\frac{\pi}{3}$ and thus $\left(\frac{3}{2}, \pm\frac{\pi}{3}\right)$. **M1A1**

$r = a + 2\cos\theta$ is symmetrical about the initial line which it intercepts at $\left(\frac{5}{2}, 0\right)$ and has a cusp at $\left(0, \pm\cos^{-1}\left(-\frac{1}{4}\right)\right)$. It passes through $\left(\frac{1}{2}, \pm\frac{\pi}{2}\right)$ and only exists for

$-\cos^{-1}\left(-\frac{1}{4}\right) < \theta < \cos^{-1}\left(-\frac{1}{4}\right)$.

$r = 2 + \cos 2\theta$ is symmetrical about both the initial line, and its perpendicular. It passes through

$(3,0)$, $(3,\pi)$, and $\left(1, \pm\frac{\pi}{2}\right)$

Sketch **G6 (8)**

(iii) If $a = 1$, then the curves meet where $2\cos^2\theta - 2\cos\theta = 0$, i.e. $\cos\theta = 1$ at $(3,0)$ where they touch, and $\cos\theta = 0$ at $\left(1, \pm\frac{\pi}{2}\right)$

$r = a + 2\cos\theta$ is symmetrical about the initial line which it intercepts at $(3, 0)$ and has a cusp at $\left(0, \pm\cos^{-1}\left(-\frac{1}{2}\right)\right) = \left(0, \pm\frac{2\pi}{3}\right)$. It passes through $\left(1, \pm\frac{\pi}{2}\right)$ and only exists for

$-\frac{2\pi}{3} < \theta < \frac{2\pi}{3}$.

Sketch **G3**

If $a = 5$, then the curves meet where $2\cos^2\theta - 2\cos\theta - 4 = 0$, i.e. only $\cos\theta = -1$ at $(3,\pi)$ where they touch, as $\cos\theta \neq 2$.

$r = a + 2\cos\theta$ is symmetrical about the initial line which it intercepts at $(7, 0)$ and $(3, \pi)$. It also passes through $\left(5, \pm\frac{\pi}{2}\right)$.

Sketch **G3 (6)**

6. (i)

$$f_\alpha(x) = \tan^{-1}\left(\frac{x\tan\alpha+1}{\tan\alpha-x}\right)$$

$$f'_\alpha(x) = \frac{1}{1+\left(\frac{x\tan\alpha+1}{\tan\alpha-x}\right)^2}\frac{(\tan\alpha-x)\tan\alpha+(x\tan\alpha+1)}{(\tan\alpha-x)^2}$$

M1 A1

$$= \frac{\tan^2\alpha+1}{(\tan\alpha-x)^2+(x\tan\alpha+1)^2}$$

$$= \frac{\sec^2\alpha}{\tan^2\alpha+x^2+x^2\tan^2\alpha+1} = \frac{\sec^2\alpha}{\sec^2\alpha\,(1+x^2)} = \frac{1}{1+x^2}$$

M1 **M1** ***A1 (5)**

as required.

Alternative

$$f_\alpha(x) = \tan^{-1}\left(\frac{x\tan\alpha+1}{\tan\alpha-x}\right)$$

$$= \tan^{-1}\left(\frac{x+\cot\alpha}{1-x\cot\alpha}\right)$$

$$= \tan^{-1}\left(\frac{\tan(\tan^{-1}x)+\tan\left(\frac{\pi}{2}-\alpha\right)}{1-\tan(\tan^{-1}x)\tan\left(\frac{\pi}{2}-\alpha\right)}\right)$$

M1 A1

$$= \tan^{-1}\left(\tan\left(\tan^{-1}x+\frac{\pi}{2}-\alpha\right)\right)$$

M1

$= \tan^{-1}x+\frac{\pi}{2}-\alpha$ if this is less than $\frac{\pi}{2}$, i.e. if $x<\tan\alpha$

or $= \tan^{-1}x-\frac{\pi}{2}-\alpha$ if $x>\tan\alpha$ **M1**

So $f'_\alpha(x) = \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ ***A1 (5)**

Thus $f_\alpha(x) = \tan^{-1}x+c$

$$f_\alpha(0) = \tan^{-1}\left(\frac{1}{\tan\alpha}\right) = \tan^{-1}(\cot\alpha) = \frac{\pi}{2}-\alpha$$

$f_\alpha(x) = 0$ when $x = -\cot\alpha$

There is a discontinuity at $x = \tan\alpha$, with $f_\alpha(x)$ approaching $\frac{\pi}{2}$ from below and $-\frac{\pi}{2}$ from above.

As $x \to \pm\infty$, $f_\alpha(x) \to \tan^{-1}(-\tan\alpha) = -\alpha$

So $f_\alpha(x) = \tan^{-1} x + \frac{\pi}{2} - \alpha$ for $x < \tan\alpha$ and $f_\alpha(x) = \tan^{-1} x - \frac{\pi}{2} - \alpha$ for $x > \tan\alpha$

Sketch **G1 G1 G1 (3)**

$$y = f_\alpha(x) - f_\beta(x) =$$

$\left(\frac{\pi}{2} - \alpha\right) - \left(\frac{\pi}{2} - \beta\right) = \beta - \alpha$ for $x < \tan\alpha$

$\left(-\frac{\pi}{2} - \alpha\right) - \left(\frac{\pi}{2} - \beta\right) = \beta - \alpha - \pi$ for $\tan\alpha < x < \tan\beta$

and $\left(-\frac{\pi}{2} - \alpha\right) - \left(-\frac{\pi}{2} - \beta\right) = \beta - \alpha$ for $x > \tan\beta$

Sketch **G1 G1 G1 (3)**

(ii) $g(x) = \tanh^{-1}(\sin x) - \sinh^{-1}(\tan x)$

$$g'(x) = \frac{1}{1 - \sin^2 x}\cos x - \frac{1}{\sqrt{1 + \tan^2 x}}\sec^2 x$$

M1 A1 **A1**

$$= \frac{\cos x}{\cos^2 x} - \frac{\sec^2 x}{|\sec x|} = \sec x - \frac{\sec^2 x}{-\sec x} = 2\sec x$$

M1 ***A1 (5)**

as required, for $\sec x < 0$, i.e. for $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

(For $\sec x > 0$, $g'(x) = 0$)

Sketch **G1 G1 G1 G1 (4)**

7.

$$z = \frac{e^{i\theta} + e^{i\varphi}}{e^{i\theta} - e^{i\varphi}}$$

$$= \frac{\cos\theta + i\sin\theta + \cos\varphi + i\sin\varphi}{\cos\theta + i\sin\theta - \cos\varphi - i\sin\varphi}$$

M1

$$= \frac{2\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2} + 2i\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}}{-2\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2} + 2i\cos\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}}$$

M1 A1 A1

$$= \frac{2\cos\frac{\theta-\varphi}{2}\left(\cos\frac{\theta+\varphi}{2} + i\sin\frac{\theta+\varphi}{2}\right)}{2\sin\frac{\theta-\varphi}{2}\left(i\cos\frac{\theta+\varphi}{2} - \sin\frac{\theta+\varphi}{2}\right)}$$

$$= -i\cot\frac{\theta-\varphi}{2}$$

$$= i\cot\frac{\varphi-\theta}{2}$$

***A1 (5)**

as required.

Alternatively,

$$z = \frac{e^{i\theta} + e^{i\varphi}}{e^{i\theta} - e^{i\varphi}} = \frac{e^{i\left(\frac{\theta-\varphi}{2}\right)} + e^{-i\left(\frac{\theta-\varphi}{2}\right)}}{e^{i\left(\frac{\theta-\varphi}{2}\right)} - e^{-i\left(\frac{\theta-\varphi}{2}\right)}} = \frac{2\cos\frac{\theta-\varphi}{2}}{2i\sin\frac{\theta-\varphi}{2}} = -i\cot\frac{\theta-\varphi}{2} = i\cot\frac{\varphi-\theta}{2}$$

M1 **M1 A1 A1** ***A1 (5)**

$$|z| = \left|\cot\frac{\theta-\varphi}{2}\right|$$

M1 A1

$$|\arg z| = \frac{\pi}{2}$$

[or $\arg z = \frac{\pi}{2}$ or $\frac{3\pi}{2}$]

M1 A1 (4)

(ii) Let $a = e^{i\alpha}$ and $b = e^{i\beta}$ **M1** then $x = a + b = e^{i\alpha} + e^{i\beta}$ and $AB = b - a = e^{i\beta} - e^{i\alpha}$

$$\arg x - \arg AB = \arg\frac{x}{AB} = \arg\frac{e^{i\alpha} + e^{i\beta}}{e^{i\beta} - e^{i\alpha}}$$

so using (i), $|\arg x - \arg AB| = \frac{\pi}{2}$ **A1** and thus OX and AB are perpendicular, since $x = a + b \neq 0$ and $a \neq b$ as A and B are distinct. **E1 (3)**

Alternative:- $0, a, a+b, b$ define a rhombus OAXB as $|a| = |b| = 1$. Diagonals of a rhombus are perpendicular (and bisect one another).

(iii) $h = a + b + c$ so $AH = a + b + c - a = b + c$ and $BC = c - b$ and thus

$$\frac{AH}{BC} = \frac{b+c}{c-b}$$

B1

as $c - b \neq 0$

From (ii),

$$\left|\arg\frac{AH}{BC}\right| = \frac{\pi}{2}$$

so BC is perpendicular to AH **E1**

unless $b + c = 0$ **E1** in which case $h = a$ **E1 (4)**

(iv) $p = a + b + c \quad q = b + c + d \quad r = c + d + a \quad s = d + a + b$

The midpoint of AQ is $\frac{a+q}{2} = \frac{a+b+c+d}{2}$ and so by its symmetry it is also the midpoint of BR, CS, and DP, **B1 E1**

and thus ABCD is transformed to PQRS by a rotation of π radians about midpoint of AQ. **E1 B1 (4)**

Alternatively, ABCD is transformed to PQRS by an enlargement scale factor -1 , centre of enlargement midpoint of AQ.

8. (i) Suppose $x_k \geq 2 + 4^{k-1}(a-2)$ for some particular integer k (and this is positive as $a > 2$)

E1

Then $x_{k+1} = {x_k}^2 - 2 \geq [2 + 4^{k-1}(a-2)]^2 - 2 = 4 + 4^k(a-2) + 4^{2k-2}(a-2)^2 - 2$

$$= 2 + 4^k(a-2) + 4^{2k-2}(a-2)^2$$

$$> 2 + 4^k(a-2)$$

M1 A1

which is the required result for $k+1$.

For $n = 1$, $2 + 4^{n-1}(a-2) = 2 + a - 2 = a$ so in this case, $x_n = 2 + 4^{n-1}(a-2)$ **B1** and thus by induction $x_n \geq 2 + 4^{n-1}(a-2)$ for positive integer n. **E1 (5)**

(ii) If $|x_k| \leq 2$, then $0 \leq |x_k|^2 \leq 4$, so $-2 \leq |x_k|^2 - 2 \leq 2$, that is $-2 \leq x_{k+1} \leq 2$. **M1A1**

If $|a| \leq 2$, $|x_1| \leq 2$ and thus by induction $-2 \leq x_n \leq 2$, that is $x_n \nrightarrow \infty$ **E1**

Whether $a = \pm\alpha$, x_2 would equal the same value, namely $\alpha^2 - 2$. **E1**

So to consider $|a| \geq 2$, we only need consider $a > 2$ to discuss the behaviour of all terms after the first. Therefore, from part (i), we know $x_n \geq 2 + 4^{n-1}(|a|-2)$ for $n \geq 2$, and thus $x_n \to \infty$ as $n \to \infty$; **B1** hence we have shown $x_n \to \infty$ as $n \to \infty$ if and only if $|a| \geq 2$. **(5)**

(iii)

$$y_k = \frac{Ax_1x_2\cdots x_k}{x_{k+1}}$$

$$y_{k+1} = \frac{Ax_1x_2\cdots x_{k+1}}{x_{k+2}} = \frac{{x_{k+1}}^2}{x_{k+2}}y_k$$

M1

Suppose that

$$y_k = \frac{\sqrt{{x_{k+1}}^2 - 4}}{x_{k+1}}$$

for some positive integer k, **E1** then

$$y_{k+1} = \frac{{x_{k+1}}^2}{x_{k+2}}\frac{\sqrt{{x_{k+1}}^2 - 4}}{x_{k+1}} = \frac{x_{k+1}\sqrt{{x_{k+1}}^2 - 4}}{x_{k+2}}$$

As $x_{k+2} = {x_{k+1}}^2 - 2$, $x_{k+1} = \sqrt{x_{k+2} + 2}$, and $\sqrt{{x_{k+1}}^2 - 4} = \sqrt{x_{k+2} - 2}$,

and thus,

$$y_{k+1} = \frac{\sqrt{x_{k+2} + 2}\sqrt{x_{k+2} - 2}}{x_{k+2}} = \frac{\sqrt{{x_{k+2}}^2 - 4}}{x_{k+2}}$$

M1 A1

which is the required result for $k+1$.

$$y_1 = \frac{Ax_1}{x_2}$$

and also we wish to have

$$y_1 = \frac{\sqrt{{x_2}^2 - 4}}{x_2}$$

M1

then $Ax_1 = \sqrt{{x_2}^2 - 4}$, that is $A^2{x_1}^2 = {x_2}^2 - 4$, and as $x_1 = a$, $x_2 = {x_1}^2 - 2 = a^2 - 2$

so

$A^2a^2 = (a^2 - 2)^2 - 4 = a^4 - 4a^2$, $A^2 = a^2 - 4$, and thus $a = \sqrt{A^2 + 4}$, as $a \neq 0$ nor $-\sqrt{A^2 + 4}$ because $a > 2$. **A1 E1**

So as the result is true for y_1 , and we have shown it to be true for y_{k+1} if it is true for y_k , it is true by induction for all positive integer n that

$$y_n = \frac{\sqrt{{x_{n+1}}^2 - 4}}{x_{n+1}}$$

E1 (8)

As $a > 2$ from (ii) $x_n \to \infty$ as $n \to \infty$ **M1** and thus using result just proved, $y_n \to 1$ as $n \to \infty$, i.e. the sequence converges. ***A1 (2)**

9.

Using the sine rule, from triangle PQR

$$\frac{PR}{\sin\theta} = \frac{PQ}{\sin\left(\frac{2\pi}{3} - \varphi\right)}$$

M1 A1

From triangle PQC

$$\frac{PQ}{\sin\frac{\pi}{3}} = \frac{a - x}{\sin\left(\frac{2\pi}{3} - \theta\right)}$$

A1

From triangle PBR

$$\frac{PR}{\sin\frac{\pi}{3}} = \frac{x}{\sin\varphi}$$

A1

Eliminating PR and PQ between these three equations

$$x \sin\frac{\pi}{3} \sin\left(\frac{2\pi}{3} - \varphi\right) \sin\left(\frac{2\pi}{3} - \theta\right) = \sin\varphi \ \sin\theta\,(a - x) \sin\frac{\pi}{3}$$

M1 A1

Hence

$$x\left(\frac{\sqrt{3}}{2}\cos\varphi + \frac{1}{2}\sin\varphi\right)\left(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right) = (a - x)\sin\varphi\ \sin\theta$$

giving

$$(\sqrt{3}\ \cot\varphi + 1)(\sqrt{3}\ \cot\theta + 1)x = 4(a - x)$$

as required. **M1 *A1 (8)**

If the ball has speed v_1 moving from P to Q, speed v_2 moving from Q to R, and speed v_3 moving from R to P,

then CLM at Q parallel to CA gives $v_1 \cos\left(\frac{2\pi}{3} - \theta\right) = v_2 \cos\frac{\pi}{3}$ and NELI perpendicular to CA gives $e v_1 \sin\left(\frac{2\pi}{3} - \theta\right) = v_2 \sin\frac{\pi}{3}$, and dividing these gives $e \tan\left(\frac{2\pi}{3} - \theta\right) = \tan\frac{\pi}{3}$

M1 A1

and similarly,

CLM at R parallel to AB gives $v_2 \cos\frac{\pi}{3} = v_3 \cos\varphi$ and NELI perpendicular to AB gives

$e v_2 \sin\frac{\pi}{3} = v_3 \sin\varphi$, and dividing these gives $e \tan\frac{\pi}{3} = \tan\varphi$. **A1**

$e \tan\left(\frac{2\pi}{3} - \theta\right) = \tan\frac{\pi}{3}$ yields $e\frac{-\sqrt{3} - \tan\theta}{1 - \sqrt{3}\tan\theta} = \sqrt{3}$ **M1** which simplifies to

$e\left(\sqrt{3}+\tan\theta\right) = \sqrt{3}\left(\sqrt{3}\tan\theta - 1\right)$, or in turn, $(3-e)\tan\theta = \sqrt{3}(1+e)$ and so $\cot\theta = \frac{(3-e)}{\sqrt{3}(1+e)}$ **A1**

$e\tan\frac{\pi}{3} = \tan\varphi$ yields $\cot\varphi = \frac{1}{e\sqrt{3}}$ **A1**

Substituting these two expressions into the first result of the question,

$$\left(\frac{1}{e}+1\right)\left(\frac{(3-e)}{(1+e)}+1\right)x = 4(a-x)$$

M1

This simplifies to

$$x\,\frac{1+e}{e}\,\frac{4}{1+e} = 4(a-x)$$

that is

$$x = e(a-x)$$

so

$$x = \frac{ae}{1+e}$$

as required. ***A1 (8)**

To continue the motion at P, then similarly to before, the third impact gives $e\tan\left(\frac{2\pi}{3}-\varphi\right) = \tan\theta$

M1

So

$$\tan\theta = e\frac{-\sqrt{3}-\tan\varphi}{1-\sqrt{3}\tan\varphi} = e\frac{\sqrt{3}(e+1)}{3e-1}$$

and thus, using the previously found result for $\cot\theta$

$$\frac{(3-e)}{\sqrt{3}(1+e)} = \frac{3e-1}{\sqrt{3}(e+1)e}$$

M1 A1

That is $e(3-e) = 3e-1$, that is $e^2 = 1$ and as $e \geq 0$, $e = 1$ (and not -1) ***B1 (4)**

10. (i) At time t, the point where the string is tangential to the cylinder, **M1** say T is at $(a\cos\theta, a\sin\theta)$, **A1** the piece of string that remains straight is of length $b - a\theta$, **M1** , the vector representing the string is thus $(b - a\theta)\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$ **dM1 A1** so the particle is at the point $(a\cos\theta - (b - a\theta)\sin\theta, a\sin\theta + (b - a\theta)\cos\theta)$. **M1 A1 (7)**

$$\dot{x} = -a\dot{\theta}\sin\theta - (b - a\theta)\dot{\theta}\cos\theta + a\dot{\theta}\sin\theta = -(b - a\theta)\dot{\theta}\cos\theta$$

$$\dot{y} = a\dot{\theta}\cos\theta - (b - a\theta)\dot{\theta}\sin\theta - a\dot{\theta}\cos\theta = -(b - a\theta)\dot{\theta}\sin\theta$$

M1 A1

Thus the speed is $\sqrt{\left((b - a\theta)\dot{\theta}\cos\theta\right)^2 + \left((b - a\theta)\dot{\theta}\sin\theta\right)^2} = (b - a\theta)\dot{\theta}$ as required. **M1 A1 (4)**

(ii) The only horizontal force on the particle is the tension in the string, which is perpendicular to the velocity at any time, so kinetic energy is conserved. **E1** Therefore,

$$\frac{1}{2}m\left((b - a\theta)\dot{\theta}\right)^2 = \frac{1}{2}mu^2$$

M1

and so, as $(b - a\theta)\dot{\theta}$ and u are both positive $(b - a\theta)\dot{\theta} = u$ ***A1 (3)**

(iii) The tension in the string, using instantaneous circular motion, at time t is

$$\frac{mu^2}{(b - a\theta)}$$

M1 A1

As $(b - a\theta)\dot{\theta} = u$, integrating with respect to t,

$$b\theta - \frac{a\theta^2}{2} = ut + c$$

M1

but when $t = 0$, $\theta = 0$ so $c = 0$. **M1 A1**

Thus, $b\theta - \frac{a\theta^2}{2} = ut$

i.e.

$$\theta^2 - \frac{2b\theta}{a} + \frac{b^2}{a^2} = \frac{b^2}{a^2} - \frac{2ut}{a} = \frac{b^2 - 2aut}{a^2}$$

Alternatively, integrating $(b - a\theta)\dot{\theta} = u$ with respect to t,

$$-\frac{(b - a\theta)^2}{2a} = ut + k$$

M1

When $t = 0$, $\theta = 0$ so $k = -\frac{b^2}{2a}$ **M1 A1**

$$\frac{(b - a\theta)^2}{2a} = \frac{b^2}{2a} - ut = \frac{b^2 - 2aut}{2a}$$

Thus, taking positive roots,

$$\frac{b - a\theta}{a} = \frac{\sqrt{b^2 - 2aut}}{a}$$

Hence, the tension is

$$\frac{mu^2}{\sqrt{b^2 - 2aut}}$$

***A1 (6)**

11. (i)

$$P(Y=n)=P(n\le X<n+1)=\int_n^{n+1}\lambda\,e^{-\lambda x}\,dx=\left[-\,e^{-\lambda x}\right]_n^{n+1}=-\,e^{-\lambda(n+1)}+\,e^{-\lambda n}$$

M1 **M1**

$$=\left(1-e^{-\lambda}\right)e^{-\lambda n}$$

as required. ***A1 (3)**

(ii)

$$P(Z<z)=\sum_{r=0}^{\infty}P(r\le X<r+z)=\sum_{r=0}^{\infty}\int_r^{r+z}\lambda\,e^{-\lambda x}\,dx=\sum_{r=0}^{\infty}\left[-\,e^{-\lambda x}\right]_r^{r+z}$$

M1 **M1**

$$=\sum_{r=0}^{\infty}\left(-\,e^{-\lambda(r+x)}+\,e^{-\lambda r}\right)=\sum_{r=0}^{\infty}\left(1-e^{-\lambda z}\right)e^{-\lambda r}$$

M1 A1

$$=\left(1-e^{-\lambda z}\right)\frac{1}{1-e^{-\lambda}}=\frac{1-e^{-\lambda z}}{1-e^{-\lambda}}$$

using sum of an infinite GP with magnitude of common ratio less than one.

M1 *A1 (6)

(iii) As $P(Z<z)=\frac{1-e^{-\lambda z}}{1-e^{-\lambda}}$, $f_Z(z)=\frac{d}{dz}\left(\frac{1-e^{-\lambda z}}{1-e^{-\lambda}}\right)=\frac{\lambda e^{-\lambda z}}{1-e^{-\lambda}}$ **M1**

so

$$E(Z)=\int_0^1 z\,\frac{\lambda e^{-\lambda z}}{1-e^{-\lambda}}\,dz=\frac{1}{1-e^{-\lambda}}\left\{\left[-ze^{-\lambda z}\right]_0^1+\int_0^1 e^{-\lambda z}\,dz\right\}$$

M1 **M1**

$$=\frac{1}{1-e^{-\lambda}}\left\{-e^{-\lambda}-\left[\frac{e^{-\lambda z}}{\lambda}\right]_0^1\right\}=\frac{1}{1-e^{-\lambda}}\left\{-e^{-\lambda}-\frac{e^{-\lambda}}{\lambda}+\frac{1}{\lambda}\right\}$$

A1

$$=\frac{1}{\lambda}-\frac{e^{-\lambda}}{1-e^{-\lambda}}$$

or alternatively

$$\frac{1}{\lambda}\,\frac{\left(1-(\lambda+1)e^{-\lambda}\right)}{1-e^{-\lambda}}$$

A1 (5)

(iv)

$$P(Y = n \; and \; z_1 < Z < z_2) = P(n + z_1 < X < n + z_2)$$

$$= \int_{n+z_1}^{n+z_2} \lambda \, e^{-\lambda x} \, dx = \left[- e^{-\lambda x}\right]_{n+z_1}^{n+z_2} = - e^{-\lambda(n+z_2)} + e^{-\lambda(n+z_1)} = e^{-\lambda n}\left(e^{-\lambda z_1} - e^{-\lambda z_2}\right)$$

M1 **A1**

$$P(Y = n \; and \; z_1 < Z < z_2) = e^{-\lambda n}\left(e^{-\lambda z_1} - e^{-\lambda z_2}\right)$$

$$= \left(1 - e^{-\lambda}\right)e^{-\lambda n}\left(\frac{1 - e^{-\lambda z_2}}{1 - e^{-\lambda}} - \frac{1 - e^{-\lambda z_1}}{1 - e^{-\lambda}}\right)$$

M1 A1

$$= P(Y = n) \times P(\, z_1 < Z < z_2)$$ **M1**

so Y and Z are independent. **E1** **(6)**

12. (i)

$$P(X_{12}=1)=\frac{1}{6}\,,P(X_{12}=0)=\frac{5}{6}\,,P(X_{23}=1)=\frac{1}{6}\,,P(X_{23}=0)=\frac{5}{6}$$

If $X_{23}=1$, then players 2 and 3 score the same as one another. In that case, $X_{12}=1$ would mean that player 1 also obtained that same score so $P(X_{12}=1|X_{23}=1)=\frac{1}{6}=P(X_{12}=1)$.

If $X_{23}=1$, $X_{12}=0$ would mean that player 1 obtained a different score so

$$P(X_{12}=0|X_{23}=1)=\frac{5}{6}=P(X_{12}=0)$$

If $X_{23}=0$, then players 2 and 3 score differently to one another. In that case, $X_{12}=1$ would mean that player 1 also obtained the same score as player 2 so $P(X_{12}=1|X_{23}=0)=\frac{1}{6}=P(X_{12}=1)$

If $X_{23}=0$, $X_{12}=0$ would mean that player 1 obtained a different score to player 2 so

$$P(X_{12}=0|X_{23}=0)=\frac{5}{6}=P(X_{12}=0)$$

Hence X_{12} is independent of X_{23}. **M1 A1 (2)**

Alternatively,

X_{12} X_{23}

1 1 requires players 2 and 3 to both score same as player 1 so

$$P(X_{12}=1\ and\ X_{23}=1)=\frac{1}{36}=\frac{1}{6}\times\frac{1}{6}=P(X_{12}=1)\times P(X_{23}=1)$$

1 0 requires player 2 to score the same as player as player 1, and player 3 score differently so

$$P(X_{12}=1\ and\ X_{23}=0)=\frac{5}{36}=\frac{1}{6}\times\frac{5}{6}=P(X_{12}=1)\times P(X_{23}=0)$$

0 1 requires players 2 and 3 to score the same as one another, and player 1 score differently so

$$P(X_{12}=0\ and\ X_{23}=1)=\frac{5}{36}=\frac{5}{6}\times\frac{1}{6}=P(X_{12}=0)\times P(X_{23}=1)$$

0 0 requires both player 1 and 3 to score differently to player 2 so

$$P(X_{12}=0\ and\ X_{23}=0)=\frac{25}{36}=\frac{5}{6}\times\frac{5}{6}=P(X_{12}=0)\times P(X_{23}=0)$$

Hence X_{12} is independent of X_{23}. **M1 A1 (2)**

If total score is T , then

$$T=\sum_{i<j}X_{ij}$$

M1

so

$$E(T) = E\left(\sum_{i<j} X_{ij}\right) = \sum_{i<j} E(X_{ij}) = {}^{n}C_2\ E(X_{12}) = {}^{n}C_2\ \left(1 \times \frac{1}{6} + 0 \times \frac{5}{6}\right) = \frac{n(n-1)}{12}$$

M1 **A1**

$$Var(T) = Var\left(\sum_{i<j} X_{ij}\right) = \sum_{i<j} Var(X_{ij}) = {}^{n}C_2\, Var(X_{12}) = {}^{n}C_2\ \left(1^2 \times \frac{1}{6} + 0^2 \times \frac{5}{6} - \frac{1}{6}^2\right)$$

M1

$$= \frac{5n(n-1)}{72}$$

A1 (5)

(ii)

$$Var(Y_1 + Y_2 + \ \dots + Y_m) = E((Y_1 + Y_2 + \ \dots + Y_m)^2) - [E(Y_1 + Y_2 + \ \dots + Y_m)]^2$$

$$= E\left({Y_1}^2 + {Y_2}^2 + \ \dots + {Y_m}^2 + 2Y_1Y_2 + 2Y_1Y_3 + \ \dots + 2Y_{n-1}Y_n\right) - [E(Y_1) + E(Y_2) + \ \dots + E(Y_m)]^2$$

$$= E\left(\sum_{i=1}^{m} {Y_i}^2\right) + 2E\left(\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} Y_iY_j\right) - (0 + 0 + \cdots + 0)^2$$

$$= \sum_{i=1}^{m} E\left({Y_i}^2\right) + 2\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} E\left(Y_iY_j\right)$$

M1 *A1 (2)

(iii)

$$P(Z_{12} = 1) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

If $Z_{23} = 1$ then player 2 has rolled an even score and player 3 has scored the same so, in this case, for $Z_{12} = 1$, require player 1 to roll the score that player has so $P(Z_{12} = 1|Z_{23} = 1) = \frac{1}{6}$.

Therefore, $P(Z_{12} = 1) \neq P(Z_{12} = 1|Z_{23} = 1)$ and thus Z_{12} and Z_{23} are not independent.

Alternatively,
$P(Z_{12} = 1) = \frac{1}{12}$, $P(Z_{23} = 1) = \frac{1}{12}$

For $Z_{12} = 1$ and $Z_{23} = 1$ we require all three players to score the same even number so

$$P(Z_{12} = 1\ and\ Z_{23} = 1\,) = \frac{3}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{72} \neq \frac{1}{12} \times \frac{1}{12} = P(Z_{12} = 1) \times P(Z_{23} = 1)$$

and thus they are not independent. **M1 A1 (2)**

Using part (ii), let $Y_1 = Z_{12}$, let $Y_2 = Z_{13}$, … let $Y_m = Z_{(n-1)n}$

(and with $m = {}^{n}C_2 = \frac{n(n-1)}{2}$).

$P(Z_{12} = 1) = \frac{1}{12}$, $P(Z_{12} = -1) = \frac{1}{12}$, $P(Z_{12} = 0) = \frac{5}{6}$ so $E(Z_{12}) = 0$ and $E\left({Z_{12}}^2\right) = \frac{1}{6}$ and likewise for all other Z (Y!). **B1** **B1**

If total score is U , then

$$U = \sum_{i<j} Z_{ij}$$

so

$$E(U) = E\left(\sum_{i<j} Z_{ij}\right) = \sum_{i<j} E\left(Z_{ij}\right) = 0$$

B1

which means we can apply the result of (ii).

If $Z_{12} = 1$ then $Z_{13} = 1$ or $Z_{13} = 0$

If $Z_{12} = -1$ then $Z_{13} = -1$ or $Z_{13} = 0$

Otherwise $Z_{12} = 0$

So $E(Z_{12}Z_{13}) = 1 \times 1 \times \frac{1}{72} + -1 \times -1 \times \frac{1}{72} = \frac{1}{36}$ **M1 A1**

So

$$Var(U) = \frac{n(n-1)}{2} \times \frac{1}{6} + 2 \times n \times {}^{n-1}C_2 \times \frac{1}{36} = \frac{n(n-1)}{12} + \frac{n(n-1)(n-2)}{36}$$

M1 **M1 A1**

$$= \frac{n(n-1)}{36}\big(3 + (n-2)\big) = \frac{n(n-1)(n+1)}{36} = \frac{n(n^2-1)}{36}$$

***A1 (9)**

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