Sixth Term Examination Paper [STEP]

Mathematics 2 [9470]

2021

Examiner's Report

Mark Scheme

STEP MATHEMATICS 2

2021

Examiner's Report

STEP 2 2021

Introduction

Candidates were generally well prepared for many of the questions on this paper, with the questions requiring more standard operations seeing the greatest levels of success. Candidates need to ensure that solutions to the questions are supported by sufficient evidence of the mathematical steps, for example when proving a given result or deducing the properties of graphs that are to be sketched.

In a significant number of steps there were marks lost through simple errors such as mistakes in arithmetic or confusion of sine and cosine functions, so it is important for candidates to maintain accuracy in their solutions to these questions.

Question 1

Many candidates were able to prove the given identity in the opening sentence of the question, although there were a large number of attempts that took the approach of expressing both the left and right sides of the identity in terms of $\cos a$ and $\sin a$. Those who recognised that the result followed quickly from applying the identity for $\cos(A \pm B)$ for $A = 3a$ and $B = a$ were then much more able to find the similar identity for $\sin a \cos 3a$.

Many candidates were able to apply the identity from the start of the question to the equation in part (i) and went on to solve the equation successfully in the required interval. A small number of candidates did not realise that it was not necessary to express the equation as a polynomial in $\cos a$ and so encountered a more difficult polynomial to solve in order to reach the solutions. In many cases this did not result in the correct set of solutions being found.

Part (ii) was less well attempted in general. While almost all candidates realised that writing the tangent functions in terms of sine and cosine would be useful many were not able to rearrange into a sufficiently useful form to make further progress on the question. Those who did often managed to reach the required result without too much difficulty. Candidates had little problem finding the full set of solutions to the two equations deduced in the first section of (ii), but then most failed to realise that some of those solutions were not possible as the equation involved tangent functions.

Question 2

On the whole, candidates performed well on this question. Almost all attempts correctly verified the identity in (i). Part (ii) however received more poor attempts than any other part. Candidates who understood what was being asked of them almost always scored all the marks, whilst those who misunderstood the meaning of the question often scored 0. The most common mistakes were to assume already that $p = a + b$ and $q = a^2 + b^2$, which is what the question required them to show, or to try to evaluate the discriminant of the quadratic in attempt to show it had real roots; these candidates failed to realise that the roots could be complex, as indicated by the first line of the question. Some candidates failed to sufficiently justify why the relation for r held, not realising that they had to show the opposite implication to what they had done in (i).

Part (iii) had more successful attempts than (ii). The most common mistake was to not use part (i), as the question specified, to prove that c was a root, and instead to expand out every term in terms of a, b, c; such attempts could not score credit for showing c was a root. Those that spotted how to use the relation in (i) would give short, quick solutions. Many candidates however were able to deduce the last relation between s, t, u, v even if they were unable to successfully answer earlier parts of the question, spotting that the product of the roots should give rise to a multiple of the constant term in the cubic. Again, some candidates once again expanded everything in terms of a, b, c to verify the relation, which did not score credit as it was not deduced from the cubic.

Candidates were able to score credit in (iv) without attempting all the previous parts and many did, often successfully. The simplest solution involving the cubic in (iii) lead quickly to the values of a, b, c, although it was possible to solve the equations by substituting them into one another, which led to the same cubic expression. However, despite many attempts finding the solutions a, b, c, surprisingly few actually verified that their claimed solution did actually satisfy all four equations, leading to incomplete solutions and not receiving full credit.

Question 3

This was a popular question, attempted by a large proportion of the candidates. Candidates who were able to appreciate the method by which the integer and fractional parts could be interpreted to find the original values were able to make good progress and gain high marks with relatively short solutions. Those who did not see this could produce many pages of work without making significant progress towards a solution.

In part (i) candidates were often able to deduce the values of x and y successfully, but some did not remember that the fractional part was defined as positive in the explanation at the start of the question, meaning that they found options for the final answer.

A variety of successful methods were seen for part (ii) and there were a high proportion of perfect answers. The most successful approach was to combine the simultaneous equations to reach the given two-variable equation. Another method was to analyse the set of 8 different cases to identify the unique solution. The most common problem encountered with this approach was to fail to identify all of the possible cases.

Part (iii) was found to be difficult by many of the candidates. While many were able to find the "obvious" solution of halving the value from (ii), the complication presented by the coefficient of 2 was not appreciated by all. Those who did were often then able to earn most of the marks for this part.

Question 4

Part (i) was often successfully answers, with most candidates successfully differentiating the equation of the curve and setting equal to 0 to find the stationary points.

In part (ii) some candidates did not link the coordinates of the stationary point found in (i) to the value of a that needed to be stated. In some cases, the graph when sketched extended beyond the point identified even when it had been identified correctly. The sketches of the inverse function were generally well done, although a significant number did not appreciate that the mirror image as the curve approached its stationary point would have a gradient that tends to infinity.

In part (iii) some candidates attempted to find a form for the inverse function rather than deducing what was necessary from the information given. In most cases this was not successful, although a small number did successfully reach some of the results. Despite the fact that the question asked candidates to find a real root in the cases where one exists, some candidates did not do this and instead simply stated the number of roots.

Those candidates who were successful with (iii)(b) were then usually able to complete the rest of the question successfully.

Question 5

This proved to be a popular question. In part (i), many candidates were able to use the substitution to reduce the differential equation into a form where the variables could be separated, but a surprising number struggled with the integral that resulted from this process. A variety of approaches were successfully employed by those who were able to complete the integration, but candidates often forgot the modulus function inside the logarithm, which caused problems later in the question. A small number of candidates forgot that the constant of integration would also be multiplied by $(x - a)$ in the final step of this part of the question.

In part (ii) some candidates were unsure how to use the information given about the tangent. Those who set $a = 1$ were generally able to make good progress and many correct sketches were produced. A number of candidates assumed, without justification, that the form of f(x) would remain unchanged from part (a) to part (b).

Question 6

This was not a popular question and many of the attempts made did not score well. Part (i) was relatively successful with most candidates able to show that the perpendicular distance from O to the line segment AB must be less than R for the given constraints.

Part (ii) proved to be relatively simple for those who chose to draw a clear diagram, although some candidates chose to focus on the wrong triangle meaning that the wrong angles were used.

Part (iii) caused more difficulty and many candidates were not able to understand the significance of the phrase “much less than 1” and so candidates who made assumptions about variables tending to 0 rather than using small angle approximations often scored no marks.

Solutions in part (iv) often jumped too quickly to the result printed in the question. It is important that solutions to questions in which the result to be proved has been given contain sufficient detail to show all of the steps being taken.

Question 7

Solutions to this question often highlighted a number of issues with understanding of matrices. For example, some candidates thought that, if the product of two matrices is zero, then one of the two matrices must be zero. Similarly, some solutions treated the number 1 and the identity matrix as interchangeable. There were also many poor examples of manipulation of determinants seen.

Candidates were able to engage well with part (i), although perfect solutions to this part were uncommon. In part (ii) many candidates were able to show the given result successfully. However, a number of attempts at this part of the question made the assumption that the matrix was a rotation, even though this is not given in the question.

In part (iii) there were a number of solutions that assumed that the determinant being 1 was a sufficient condition for the matrix to represent a rotation or gave an insufficient justification that the matrix represents a rotation. Many candidates were able to deduce the angles of the rotation correctly in this part.

Question 8

Many candidates were able to complete the differentiation correctly in terms of n for part (i) of this question although in some cases the result that was given was jumped to with insufficient justification following the completion of the differentiation.

Similarly, in part (ii) many candidates chose appropriate methods to reach the required relation, but did not provide sufficient working to show that the appropriate manipulations had been carried out correctly. Candidates need to ensure that solutions to questions where the answer is given provide sufficiently detailed explanation of the steps that are taken.

In part (iii) many candidates were able to evaluate the necessary base cases for the proof by induction and provided some justification for the inductive step, although in some cases it was not sufficiently clear that the values of a_n and b_n would be integers. Many candidates were then able to demonstrate some understanding of the necessary steps for the final part, but in some cases insufficient detail was present to secure full marks.

Question 9

It was pleasing to see that many candidates chose to draw a diagram to represent the setup of the problem.

Solutions to part (i) were often good, but marks were often lost due to lack of justification, both for the triangle inequality and for reasoning involving acute angles. Candidates often also failed to equate the tension on either side of the pulleys.

Many candidates attempted part (ii) having failed to complete the previous and most were able to obtain a mark here by using the results that had been given in the previous part.

Question 10

This question was generally quite poorly attempted, with many candidates not able to understand fully the situation being studied. A large proportion of candidates only attempted the first part and were unable to earn any of the marks. Of the rest many did not progress beyond the second part, with many simply claiming incorrectly that the second derivative of x is a and the second derivative of y is $-g$.

Those who did manage to solve the early parts of the question were generally quite successful with the rest of the question which was generally very well answered.

Question 11

Some candidates misread the first part of the question and therefore attempted to solve a different question than was intended. The most common such misunderstanding that the probability P_n introduced in part (i) related specifically to a train with n seats. Where candidates did not have this problem the computations were done well.

The explanation in part (ii) was also done well by most candidates who engaged meaningfully with the question. The deduction in this part of the question caused some trouble, but many were able to successfully complete this part. In particular, the reindexing of the sum within this solution was often overlooked or poorly explained.

In part (iii) many candidates were able to identify the correct simplified form. However, there was some confusion about the difference between weak and strong induction meaning that many candidates were not able to give a satisfactory explanation of how the conclusion is drawn for the final mark in this section.

Of those who successfully completed (iii) many were able to make good progress on the final part of the question.

Question 12

Many candidates were able to reach the required probability in the first part of this question, although many ignored drawn matches instead making an argument that the probability can be found by dividing the probability that A wins this game by the probability that someone wins on this game. While this argument is possible, generally far more justification was needed than candidates provided. Those who identified the necessary sequences were able to successfully reach the result in a well-justified way.

In part (ii) a small number of candidates assumed that the number of games in a match would always be even rather than showing why this must be true. Of the other candidates, many were able to explain why this is the case. Relatively few candidates failed to spot that the games in parts (ii) and (iii) could be reduced to the same game as in (i). In (ii), many candidates attempted a combinatorial argument, but a significant number failed to observe that there are two ways to order each pair where each of the players wins one of the games.

In part (iii) most candidates were able to derive the probability of winning the bold game. Most of those who reached the end of this part used logical implications in the wrong direction, for example showing that if the player is more likely to win the cautious version, then the given inequality holds.

STEP MATHEMATICS 2

2021

Mark Scheme

1

$$\cos(3a + a) \equiv \cos 3a \cos a - \sin 3a \sin a$$ **M1**
$$\cos(3a - a) \equiv \cos 3a \cos a + \sin 3a \sin a$$
$$\cos 4a + \cos 2a \equiv 2\cos 3a \cos a$$
$$\cos a \cos 3a \equiv \frac{1}{2}(\cos 4a + \cos 2a) \quad \boldsymbol{AG}$$ **A1**
$$\sin(3a + a) \equiv \sin 3a \cos a + \cos 3a \sin a$$
$$\sin(3a - a) \equiv \sin 3a \cos a - \cos 3a \sin a$$
$$\sin 4a - \sin 2a \equiv 2\cos 3a \sin a$$
$$\sin a \cos 3a \equiv \frac{1}{2}(\sin 4a - \sin 2a)$$ **B1**

(i)

$$2\cos 2x\,(2\cos x \cos 3x) = 1$$
$$2\cos 2x\,(\cos 4x + \cos 2x) = 1$$ **M1**
$$2\cos 2x\,(2\cos^2 2x + \cos 2x - 1) = 1$$ **M1**
$$4\cos^3 2x + 2\cos^2 2x - 2\cos 2x - 1 = 0$$
$$(2\cos^2 2x - 1)(2\cos 2x + 1) = 0$$ **M1**

A1

Either $\cos^2 2x = \frac{1}{2}$:

$$2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$
$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$$ **A1**

Or $\cos 2x = -\frac{1}{2}$:

$$2x = \frac{2\pi}{3}, \frac{4\pi}{3}$$
$$x = \frac{\pi}{3}, \frac{2\pi}{3}$$ **A1**

Therefore:

$$x = \frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$$

(ii) $$2\cos x \sin 3x \equiv \sin 4x + \sin 2x$$ **B1**

$$\tan x = \tan 2x \tan 3x \tan 4x$$ **M1**

$$\sin x \cos 2x \cos 3x \cos 4x = \cos x \sin 2x \sin 3x \sin 4x$$

$$(2\sin x \cos 3x)\cos 2x \cos 4x = (2\cos x \sin 3x)\sin 2x \sin 4x$$ **M1**

$$(\sin 4x - \sin 2x)\cos 2x \cos 4x = (\sin 4x + \sin 2x)\sin 2x \sin 4x$$

$$\sin 4x\,(\cos 2x \cos 4x - \sin 2x \sin 4x) = \sin 2x\,(\cos 2x \cos 4x + \sin 2x \sin 4x)$$

$$\sin 4x \cos 6x = \sin 2x \cos 2x$$ **M1**

$$\sin 4x \cos 6x = \frac{1}{2}\sin 4x$$ **M1**

M1

$$\sin 4x\,(2\cos 6x - 1) = 0$$ **A1**

Therefore $\cos 6x = \frac{1}{2}$ or $\sin 4x = 0$. **AG**

$\cos 6x = \frac{1}{2}$:

$$6x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}$$

$$x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}$$ **A1**

$\sin 4x = 0$:

$$4x = 0, \pi, 2\pi, 3\pi, 4\pi$$

$$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$$ **A1**

$\tan x$ is undefined at $x = \frac{\pi}{2}$ **B1**

$\tan 2x$ is undefined at $x = \frac{\pi}{4}, \frac{3\pi}{4}$ **B1**

So these are not solutions of the equation.

$$x = 0, \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \pi$$

2

(i) $$3pq - p^3 = 3(a+b)(a^2+b^2) - (a+b)^3$$ **M1**
$$= 2a^3 + 2b^3$$
$$= 2r \quad \boldsymbol{AG}$$ **A1**

(ii) $$2x^2 - 2px + (p^2 - q) = 0$$

The roots of the equation a and b satisfy: **M1**
$$a + b = p$$ **B1**
$$2ab = p^2 - q$$ **B1**
$$a^2 + b^2 = (a+b)^2 - 2ab$$ **B1**
$$= p^2 - (p^2 - q) = q$$
$$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$
$$= p^3 - \frac{3}{2}(p^2 - q)p$$
$$= \frac{1}{2}(3pq - p^3) = r$$ **B1**

So the three equations hold. **E1**

(iii) $$a + b = s - c \ (= p)$$
$$a^2 + b^2 = t - c^2 \ (= q)$$
$$a^3 + b^3 = u - c^3 \ (= r)$$ **M1**

By part (i):
$$3(s-c)(t-c^2) - (s-c)^3 = 2(u - c^3)$$
$$3st - 3ct - 3c^2s + 3c^3 - s^3 + 3cs^2 - 3c^2s + c^3 = 2u - 2c^3$$ **M1**
$$6c^3 - 6sc^2 + 3(s^2 - t)c + 3st - s^3 - 2u = 0$$ **A1**

Therefore c is a root of the equation
$$6x^3 - 6sx^2 + 3(s^2 - t)x + 3st - s^3 - 2u = 0 \quad \boldsymbol{AG}$$ **E1**

The other roots are a and b. **B1**

The constant term is $-6 \times$ the product of the roots: **M1**
$$-6abc = 3st - s^3 - 2u$$
$$s^3 - 3st + 2u = 6v \quad \boldsymbol{AG}$$ **A1**

(iv) By (iii) a, b and c are the roots of **M1**
$$6x^3 - 18x^2 + 24x - 12 = 0$$ **A1**
$$6(x-1)(x^2 - 2x + 2) = 0$$ **M1**
$$1, 1+i, 1-i$$ **A1**

$1 + (1+i) + (1-i) = 3$

$1^2 + (1+i)^2 + (1-i)^2 = 1 + (1 + 2i - 1) + (1 - 2i - 1) = 1$

$1^3 + (1+i)^3 + (1-i)^3 = 1 + (-2 + 2i) + (-2 - 2i) = -3$ **B1**

$1(1+i)(1-i) = 2$

3

(i)	From the 1st eqn: $\lfloor x \rfloor = 4$ and $\{y\} = 0.9$	**B1**
	From the 2nd eqn: $\{x\} = 0.6$ and $\lfloor y \rfloor = -2$	**B1**
	Clear use of $x = \lfloor x \rfloor + \{x\}$ etc.	**M1**
	Solution is $x = 4.6$, $y = -1.1$	**A1**
	NB for candidates scoring *none* of the above marks, allow a **B1** for adding both eqns. to obtain $x + y = 3.5$	
(ii)	② + ③ – ①	**M1**
	$\Rightarrow\ y + \{y\} - \lfloor y \rfloor + z + \lfloor z \rfloor - \{z\} = 6.4$	
	$\Rightarrow\ 2\{y\} + 2\lfloor z \rfloor = 6.4$	**M1**
	$\Rightarrow \{y\} + \lfloor z \rfloor = 3.2$ ***AG*** or $\{x\} + \lfloor y \rfloor = 2.1$ or $\lfloor x \rfloor + \{z\} = 1.8$	**A1**
	Similar attempts at ① + ② – ③ $\Rightarrow \{x\} + \lfloor y \rfloor = 2.1$ and ① + ③ – ② $\Rightarrow \lfloor x \rfloor + \{z\} = 1.8$	**M1**
	The remaining two 2-variable eqns. correct	**A1**
	$\Rightarrow \{y\} = 0.2$ and $\lfloor z \rfloor = 3$	**B1**
	Also (respectively) $\{x\} = 0.1$ and $\lfloor y \rfloor = 2$ and $\lfloor x \rfloor = 1$ and $\{z\} = 0.8$	**B1**
	Solution is $x = 1.1$, $y = 2.2$, $z = 3.8$	**A1**
(iii)	From ② + ③ – ①, we now get $2\{y\} + \lfloor z \rfloor = 3.2$	**B1**
	From ① + ③ – ②, we still get $\lfloor x \rfloor + \{z\} = 1.8$	**B1**
	From ① + ② – ③, we now get $\{x\} + 2\lfloor y \rfloor = 2.1$	**B1**
	First solution follows immediately from (ii): namely, $x = 1.1$, $y = 1.1$, $z = 3.8$	**B1**
	For clear evidence that the second possibility exists	**M1**
	namely: $2\{y\} + \lfloor z \rfloor = 3.2 \Rightarrow \{y\} = 0.6$ and $\lfloor z \rfloor = 2$	**A1**
	and $\{x\} + 2\lfloor y \rfloor = 2.1 \Rightarrow \{x\} = 0.1$ and $\lfloor y \rfloor = 1$	**A1**
	NB $\lfloor x \rfloor = 1$ and $\{z\} = 0.8$ follows as before	
	Second solution is $x = 1.1$, $y = 1.6$, $z = 2.8$	**A1**

4

(i) $\frac{dy}{dx} = xe^x + e^x$ **M1**

Since $e^x > 0$ for all x, the only stationary point is when $x = -1$ **A1**
Coordinates of stationary point are $(-1, -\frac{1}{e})$

Sketch showing:
$y \to \infty$ as $x \to \infty$ and $y \to 0^-$ as $x \to -\infty$ **G1**
Curve passing through $(0,0)$ with stationary point at $(-1, -\frac{1}{e})$ indicated. **G1**

(ii) -1 **B1**

Sketch showing reflection of the correct portion of the graph in the line $y = x$. **G1**
domain $\left[-\frac{1}{e}, \infty\right)$ and range $[-1, \infty)$

(iii)
(a) $e^{-x} = 5x$
$xe^x = \frac{1}{5}$
M1

$f(x) = \frac{1}{5}$
Since $f(x) > 0$ there is only one solution **A1**

$x = g\left(\frac{1}{5}\right)$

(b) $2x \ln x + 1 = 0$
Let $u = \ln x$: **M1**

$$ue^u = -\frac{1}{2}$$ **M1**

The minimum value of $f(x)$ is $-\frac{1}{e}$ and $-\frac{1}{2} < -\frac{1}{e}$, so there are no solutions. **E1**

(c) $3x \ln x + 1 = 0$
Let $u = \ln x$:

$$ue^u = -\frac{1}{3}$$ **M1**

$-\frac{1}{e} < -\frac{1}{3} < 0$ so there are two solutions for u and the greater of the two will be when $u = g\left(-\frac{1}{3}\right)$. **E1**

$x = e^{g\left(-\frac{1}{3}\right)}$ is the larger value. **A1**

(d) $x = 3 \ln x$
Let $u = \ln x$:

$$ue^{-u} = \frac{1}{3}$$ **M1**

$(-u)e^{-u} = -\frac{1}{3}$, so (as in (c)) $g\left(-\frac{1}{3}\right)$ is the greater of the two possible values for $-u$. **M1**

Therefore $x = e^{-g\left(-\frac{1}{3}\right)}$ is the smaller value. **A1**

E1

(iv) $x \ln x = \ln 10$
Let $u = \ln x$:

$$ue^u = \ln 10$$ **M1**

$u = g(\ln 10)$

$$x = e^{g(\ln 10)}$$ **A1**

5

(i)

$$\frac{dy}{dx} = (x-a)\frac{du}{dx} + u$$ **M1** **A1**

$$(x-a)\left((x-a)\frac{du}{dx} + u\right) = (x-a)u - x$$

$$(x-a)^2\frac{du}{dx} = -x$$

$$u = \int \frac{-x}{(x-a)^2}dx = \int \frac{-(x-a)-a}{(x-a)^2}dx$$ **M1**

$$u = -\ln|x-a| + \frac{a}{x-a} + c$$ **A1**

$$y = -(x-a)\ln|x-a| + a + c(x-a)$$ **A1 (ft)**

(ii)

(a) The gradient of the line through $(1,t)$ and $(t, f(t))$ is $\frac{f(t)-t}{t-1} = f'(t)$ **M1**

Applying the result from (i), with a=1 or solving the d.e. directly:

$$f(x) = -(x-1)\ln|x-1| + 1 + c(x-1)$$ **B1 (ft)**

$f(0) = 0$, so $c = 1$ **B1 (ft)**

$$y = -(x-1)\ln|x-1| + x$$

$$\frac{dy}{dx} = -\ln|x-1|$$ **M1**

$-\ln|x-1| = 0$ when $x = 0$ only (since $x < 1$) and $y = 0$ **A1 (ft)**

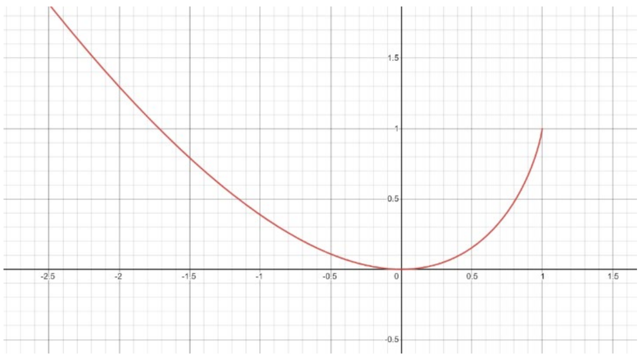
As $x \to 1^-$, $y \to 1^-$ and $\frac{dy}{dx} \to \infty$. **B1 (ft)**

Sketch showing:

Curve approaching (1,1) with a vertical tangent at that point. **G1 (ft)**

Minimum point at (0,0). **G1 (ft)**

$y \to \infty$ as $x \to -\infty$ **G1 (ft)**



(b) $f(2) = 2$, so $c = 1$ **B1 (ft)**

$$y = -(x - 1)\ln|x - 1| + x$$

$$\frac{dy}{dx} = -\ln|x - 1|$$

$-\ln|x - 1| = 0$ when $x = 2$ only (since $x > 1$) and $y = 2$. **B1 (ft)**

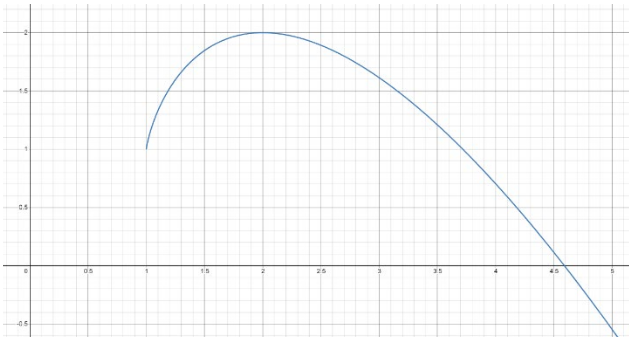
As $x \to 1^+$, $y \to 1^+$ and $\frac{dy}{dx} \to \infty$. **B1 (ft)**

Sketch showing:

Curve approaching $(1,1)$ with a vertical tangent at that point. **G1 (ft)**

Maximum point at $(2,2)$. **G1 (ft)**

The curve crossing the x-axis for some $x > 2$ and $y \to -\infty$ as $x \to \infty$ **G1 (ft)**



6

(i) The shortest distance from O to the line AB is $(R+w)\cos\alpha$ **B1**

Since $\frac{1}{3}\pi \le \alpha \le \frac{1}{2}\pi$, $0 \le \cos\alpha \le \frac{1}{2}$. **M1**

Since $w < R$, $(R+w)\cos\alpha < \frac{1}{2}(R+R) = R$, so the midpoint of the line AB lies inside the smaller circle. **E1**

(ii)

(a)
$$(R+d)^2 = (R+w)^2 + d^2 - 2d(R+w)\cos(\pi-\alpha)$$
M1 **A1**
$$R^2 + 2Rd + d^2 = R^2 + 2Rw + w^2 + d^2 + 2d(R+w)\cos\alpha$$
$$d = \frac{w(2R+w)}{2(R-(R+w)\cos\alpha)}$$
M1 **A1**

(b)
$$\angle O'AO = \alpha - \theta$$
$$\frac{\sin(\alpha-\theta)}{d} = \frac{\sin(\pi-\alpha)}{R+d}$$
M1
$$\sin(\alpha-\theta) = \frac{d\sin\alpha}{R+d}$$
A1

(iii)
$$\frac{d}{R} = \frac{\left(\frac{w}{R}\right)\left(2+\frac{w}{R}\right)}{2\left(1-\left(1+\frac{w}{R}\right)\cos\alpha\right)} \approx \frac{1}{1-\cos\alpha} \times \frac{w}{R}$$
M1 **A1**

$1-\cos\alpha > \frac{1}{2}$ and $\frac{w}{R}$ is much less than 1, so $\frac{d}{R}$ is much less than 1. **E1**

$$\sin(\alpha-\theta) = \frac{\left(\frac{d}{R}\right)\sin\alpha}{1+\left(\frac{d}{R}\right)} < \frac{d}{R}$$
M1

$\sin(\alpha-\theta)$ is much less than 1 and so $(\alpha-\theta)$ is a small angle. **M1**

Therefore $\sin(\alpha-\theta) \approx \alpha-\theta$, so $\alpha-\theta$ is much less than 1. **E1**

(iv) The longer length is $(R+w)\times 2\alpha$

The shorter length is $(R+d)\times 2\theta$
$$S = 2\alpha(R+w) - 2\theta(R+d)$$
$$S = 2(R+d+w-d)\alpha - 2(R+d)\theta$$
$$S = 2(R+d)(\alpha-\theta) + 2(w-d)\alpha$$
B1
$$\alpha-\theta \approx \frac{w\sin\alpha}{R(1-\cos\alpha)}$$
M1
$$d-w \approx \frac{\cos\alpha}{(1-\cos\alpha)} \times \frac{w}{R}$$
M1

So $S \approx 2(R+d)\frac{w\sin\alpha}{R(1-\cos\alpha)} - 2\left(\frac{\cos\alpha}{(1-\cos\alpha)} \times \frac{w}{R}\right)\alpha$

As a fraction of the longer path length:
$$\frac{S}{2\alpha(R+w)} = \frac{R+d}{R+w} \times \frac{\alpha-\theta}{\alpha} + \frac{w-d}{R+w} \approx \frac{\sin\alpha}{\alpha(1-\cos\alpha)}\frac{w}{R} - \frac{\cos\alpha}{(1-\cos\alpha)}\frac{w}{R}$$
M1
$$S \approx \left(\frac{\sin\alpha - \alpha\cos\alpha}{\alpha(1-\cos\alpha)}\right)\frac{w}{R} \quad \boldsymbol{AG}$$
A1

7

(i) $\boldsymbol{R} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ **M1**

$\boldsymbol{R} + \boldsymbol{I} = \begin{pmatrix} 1+\cos\phi & -\sin\phi \\ \sin\phi & 1+\cos\phi \end{pmatrix}$ **A1**

This must also be of the form $\begin{pmatrix} cos & -sin \\ sin & cos \end{pmatrix}$, so $(1+\cos\phi)^2 + \sin^2\phi = 1$ **M1**

$1 + 2\cos\phi = 0$

$\phi = 120° \; or \; 240°$ **A1**

In either case, three consecutive rotations is equivalent to a rotation through 0°, so

$\boldsymbol{R}^3 = \boldsymbol{I} \quad \boldsymbol{AG}$ **E1**

(ii) $\det(\boldsymbol{S}^3) = 1$

$\det(\boldsymbol{S}^3) = \det(\boldsymbol{S})^3$

Therefore $\det(\boldsymbol{S}) = 1 \quad \boldsymbol{AG}$ **B1**

$\boldsymbol{S}^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = \begin{pmatrix} a^2+bc & (a+d)b \\ (a+d)c & bc+d^2 \end{pmatrix}$

Since $\det(\boldsymbol{S}) = 1, ad - bc = 1$

$a^2 + bc = a^2 + ad - 1 = a(a+d) - 1$ **M1**

$bc + d^2 = ad + d^2 - 1 = d(a+d) - 1$

Therefore, $\boldsymbol{S}^2 = (a+d)\boldsymbol{S} - \boldsymbol{I} \quad \boldsymbol{AG}$ **A1**

$\boldsymbol{S}^3 = \boldsymbol{S}^2\boldsymbol{S} = (a+d)\boldsymbol{S}^2 - \boldsymbol{S}$ **M1**

$\boldsymbol{I} = (a+d)\big((a+d)\boldsymbol{S} - \boldsymbol{I}\big) - \boldsymbol{S}$ **M1**

$((a+d)^2 - 1)\boldsymbol{S} = (a+d+1)\boldsymbol{I}$ **A1**

If $((a+d)^2 - 1)$ and $(a+d+1)$ are non-zero, then $b = c = 0$ **B1**

In which case $ad = 1$ **M1**

since $\det(\boldsymbol{S}) = 1$ and since $\boldsymbol{S}^3 = \boldsymbol{I}$, $a = d = 1$ **A1**

But $\boldsymbol{S} \neq \boldsymbol{I}$, so this is not possible. **E1**

Therefore $a + d = -1$ **A1**

(iii) If $\boldsymbol{S} = \boldsymbol{I}$, then $\boldsymbol{S} + \boldsymbol{I} = 2\boldsymbol{I}$, which does not represent a rotation.

Therefore, the conditions of part (ii) are met and so $a + d = -1$.

Suppose that $\boldsymbol{S} + \boldsymbol{I}$ represents an anticlockwise rotation through angle θ: **M1**

$a + 1 = d + 1 = \cos\theta$

$(a+1) + (d+1) = 1$, so $a = d = -\frac{1}{2}$.

Also, $b = -c$ and $b^2 = c^2 = \frac{3}{4}$ **M1**

Therefore $\boldsymbol{S} = \begin{pmatrix} -\frac{1}{2} & \pm\frac{1}{2}\sqrt{3} \\ \mp\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}$ **A1**

Which represents a rotation through 120° or 240° **A1**

8

(i)
$$\frac{d}{dt}(t^n(1-t)^n) = nt^{n-1}(1-t)^n - nt^n(1-t)^{n-1} \quad \textbf{M1}$$

$$\frac{d^2}{dt^2}(t^n(1-t)^n) = n(n-1)t^{n-2}(1-t)^n - n^2t^{n-1}(1-t)^{n-1}$$

$$-n^2t^{n-1}(1-t)^{n-1} + n(n-1)t^n(1-t)^{n-2} \quad \textbf{A1}$$

$$= nt^{n-2}(1-t)^{n-2}[(n-1)(1-t)^2 - 2nt(1-t) + (n-1)t^2] \quad \textbf{M1}$$

$$= nt^{n-2}(1-t)^{n-2}[(4n-2)t^2 - (4n-2)t + (n-1)]$$

$$= nt^{n-2}(1-t)^{n-2}[(n-1) - 2(2n-1)t(1-t)] \quad \boldsymbol{AG} \quad \textbf{A1}$$

(ii) Integrating by parts:

$u = t^n(1-t)^n, \frac{dv}{dx} = \frac{e^t}{n!}$ **M1**

$\frac{du}{dx} = nt^{n-1}(1-t)^{n-1}(1-2t), v = \frac{e^t}{n!}$

$$T_n = \left[t^n(1-t)^n\frac{e^t}{n!}\right]_0^1 - \int_0^1 nt^{n-1}(1-t)^{n-1}(1-2t)\frac{e^t}{n!}dt$$

$$= -\int_0^1 nt^{n-1}(1-t)^{n-1}(1-2t)\frac{e^t}{n!}dt \quad \textbf{M1}$$

Integrating by parts:

$u = nt^{n-1}(1-t)^{n-1}(1-2t), \frac{dv}{dx} = \frac{e^t}{n!}$

$\frac{du}{dx} = nt^{n-2}(1-t)^{n-2}[(n-1) - 2(2n-1)t(1-t)], v = \frac{e^t}{n!}$

$$T_n = -\left[nt^{n-1}(1-t)^{n-1}\frac{e^t}{n!}\right]_0^1$$

$$+\int_0^1 nt^{n-2}(1-t)^{n-2}[(n-1) - 2(2n-1)t(1-t)]\frac{e^t}{n!}dt \quad \textbf{M1}$$

$$= \int_0^1 nt^{n-2}(1-t)^{n-2}[(n-1) - 2(2n-1)t(1-t)]\frac{e^t}{n!}dt$$

$$= \int_0^1 t^{n-2}(1-t)^{n-2}\frac{e^t}{(n-2)!} - 2(2n-1)\,t^{n-1}(1-t)^{n-1}\frac{e^t}{(n-1)!}dt \quad \textbf{M1} \quad \textbf{M1}$$

$$T_n = T_{n-2} - 2(2n-1)T_{n-1} \quad for\ n \geq 2 \quad \boldsymbol{AG} \quad \textbf{A1}$$

(iii) $$T_0 = \int_0^1 e^t \, dt = e - 1$$ **B1**

$$T_1 = \int_0^1 t(1-t)e^t \, dt$$

$$= \int_0^1 te^t - t^2e^t \, dt$$ **M1**

$$\int_0^1 te^t dt = [te^t]_0^1 - \int_0^1 e^t \, dt = 1$$

$$\int_0^1 t^2e^t dt = [t^2e^t]_0^1 - 2\int_0^1 te^t \, dt = e - 2$$ **M1**

$$T_1 = 1 - (e-2) = 3 - e$$ **A1**

T_0 and T_1 are both of the given form. **B1**

If T_{n-2} and T_{n-1} are both of the given form, then by part (ii):

$$a_n = a_{n-2} - 2(2n-1)a_{n-1}$$
$$b_n = b_{n-2} - 2(2n-1)b_{n-1}$$

If $a_{n-2}, a_{n-1}, b_{n-2}$ and b_{n-1} are all integers, so a_n and b_n will also be integers. **E1**

(iv) For $0 \le t \le 1$:

$0 \le t^n(1-t)^n \le 1$ **M1**

$0 \le e^t \le e$ **M1**

$0 \le \frac{t^n(1-t)^n}{n!}e^t \le \frac{e}{n!}$ and equality can only occur at t=0 or t=1, so $T_n > 0$ and is less than the area of a rectangle with width 1 and height $\frac{e}{n!}$.

$$0 < T_n < \frac{e}{n!}$$ **E1**

Therefore $a_n + b_ne \to 0$ as $n \to \infty$

Therefore $-\frac{a_n}{b_n} \to e$ as $n \to \infty$ **E1**

9

(i)

(a) The forces acting on the particle at P are:

$W = Mg$ (directed downwards) **M1**

$T_1 = m_1 g$ (directed towards Q) **A1**

$T_2 = m_2 g$ (directed towards R)

By the triangle inequality: **dM1**

$Mg < m_1 g + m_2 g$

$M < m_1 + m_2$ **A1**

$T_1^2 = T_2^2 + W^2 - 2T_2 W \cos\theta_2$

Since θ_2 is acute $\cos\theta_2 > 0$, so **M1**

$T_1^2 < T_2^2 + W^2$

$M^2 g^2 > m_1^2 g^2 - m_2^2 g^2$ **E1**

$\sqrt{m_1^2 - m_2^2} < M$

A1

(b) $QS = PS\tan\theta_1$ and $SR = PS\tan\theta_2$

If S divides QR in the ratio r: 1, then $QS = rSR$

$$r = \frac{\tan\theta_1}{\tan\theta_2}$$ **M1**

By the sine rule:

$$\frac{\sin\theta_2}{m_1 g} = \frac{\sin\theta_1}{m_2 g}$$ **M1**

By the cosine rule:

$$\cos\theta_1 = \frac{T_1^2 + W^2 - T_2^2}{2T_1 W} = \frac{m_1^2 + M^2 - m_2^2}{2m_1 M}$$ **M1**

Similarly:

$$\cos\theta_2 = \frac{m_2^2 + M^2 - m_1^2}{2m_2 M}$$ **M1**

Therefore:

$$r = \frac{\sin\theta_1}{\sin\theta_2} \times \frac{\cos\theta_2}{\cos\theta_1}$$

$$= \frac{m_2}{m_1} \times \frac{\frac{m_2^2 + M^2 - m_1^2}{2m_2 M}}{\frac{m_1^2 + M^2 - m_2^2}{2m_1 M}} = \frac{m_2^2 + M^2 - m_1^2}{m_1^2 + M^2 - m_2^2} \quad \boldsymbol{AG}$$ **dM1** **A1**

(ii) From the triangle of forces, the angle between T_1 and T_2 must be 90° (Pythagoras)

Therefore $\theta_1 + \theta_2 = 90°$ **B1**

By (i)(b)

$$r = \frac{m_2^2}{m_1^2}$$ **M1**

Let d be such that $QS = m_2^2 d$ and $SR = m_1^2 d$. **M1**

Since triangles PSQ and RSP are similar: **M1**

$$\frac{SP}{QS} = \frac{RS}{SP}$$ **M1**

$$PS^2 = m_1^2 m_2^2 d^2$$ **A1**

Therefore, $SP = m_1 m_2 d$ and QR= $(m_1^2 + m_2^2)d$, so the ratio of QR to SP is:

$$M^2 : m_1 m_2$$ **A1**

10

(i) To remain stationary relative to the train the bead would have to have horizontal acceleration a. **E1**

There is no horizontal force on the bead at the origin, so this is impossible. **E1**

(ii) When the particle is at the point (x, y):

Let the angle that the tangent to the curve makes with the horizontal be θ:

The wire is smooth, so gravity will be the only force with a component in the direction of the tangent to the curve.

The acceleration of the particle will be $\begin{pmatrix} \ddot{x} - a \\ \ddot{y} \end{pmatrix}$ **B1**

Therefore, resolving in the tangential direction: **M1**

$$m(\ddot{x} - a)cos\theta + m\ddot{y}sin\theta = -mgsin\theta$$ **A1**

A1

$$(\ddot{x} - a) + (\ddot{y} + g)tan\theta = 0$$

$$\dot{y} = \dot{x}tan\theta$$ **M1**

Therefore

$$\dot{x}(\ddot{x} - a) + (\ddot{y} + g)\dot{x}tan\theta = 0$$ **M1**

$$\dot{x}(\ddot{x} - a) + (\ddot{y} + g)\dot{y} = 0$$ **A1**

$$\frac{d}{dt}\left(\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy\right) = \dot{x}(\ddot{x} - a) + (\ddot{y} + g)\dot{y} = 0$$ **M1**

So the expression is constant during the motion. **A1**

(iii) Initially, $\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - ax + gy = 0$ (and throughout the motion since it is constant) **M1**

At the maximum vertical displacement $\dot{y} = 0$.

$\dot{x} = 0$ as well would only be possible at the origin (which is not maximum vertical displacement, therefore $\dot{x} = 0$ and $x \neq 0$ **M1**

Therefore, $ax = gy$

and so $g^2y^2 = a^2x^2 = a^2ky$ **M1**

Therefore, b satisfies

$$g^2b^2 = a^2kb$$

$$b = \frac{a^2k}{g^2}$$ **A1**

(iv) The square of the speed relative to the train is

$\dot{x}^2 + \dot{y}^2 = 2(ax + gy)$ **M1**

$$2\left(ax - \frac{gx^2}{k}\right)$$

$$-\frac{2g}{k}\left(x - \frac{ak}{2g}\right)^2 + \frac{a^2k}{2g}$$ **M1**

A1

Maximum speed is $a\sqrt{\frac{k}{2g}}$ **A1**

When $x = \frac{ak}{2g}$ **A1**

11

(i) $P_2 = \frac{1}{2}$ **B1**

T_3 can sit in seat S_3 if **M1**
T_1 chooses seat S_2, then T_2 chooses seat S_1
or T_1 chooses seat S_1
$P_3 = \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{2}$ **A1**

(ii) If passenger T_1 sits in seat S_k $(1 < k < n)$ then passengers T_2 to T_{k-1} all sit in their allocated seats. **E1**
The situation just before T_k arrives is then the same as for a train that did not have the $(k-1)$ seats that have been taken and for which T_k had been allocated seat S_1 **E1**

T_1 sits in seat S_1 with probability $\frac{1}{n}$, after which all the remaining passengers will get their allocated seats.

$$P(T_1 \; sits \; in \; S_1 \cap T_n \; sits \; in \; S_n) = \frac{1}{n}$$ **M1**

For $1 < k < n$, T_1 sits in seat S_k with probability $\frac{1}{n}$, so

$$P(T_1 \; sits \; in \; S_k \cap T_n \; sits \; in \; S_n) = \frac{1}{n} P_{n-k+1}$$ **M1**

If T_1 sits in S_n then it will not be possible for T_n to sit in S_n

$$P_n = \frac{1}{n} + \sum_{k=2}^{n-1} \frac{1}{n} P_{n-k+1} = \frac{1}{n}\left(1 + \sum_{r=2}^{n-1} P_r\right) \quad \boldsymbol{AG}$$ **A1**

(iii) $P_n = \frac{1}{2}$ **B1**
Case where $n = 1$ is shown in part (i)
Suppose $P_k = \frac{1}{2}$ for $1 \le k < n$: **M1**
$P_n = \frac{1}{n}\left(1 + (n-2) \times \frac{1}{2}\right) = \frac{1}{2}$ **A1**
Therefore, by induction $P_n = \frac{1}{2}$ **E1**

(iv) $Q_2 = \frac{1}{2}$ **B1**
For $n > 2$:
For $1 < k < n-1$:

$$P(T_{n-1} \; sits \; in \; S_{n-1} | T_1 \; sits \; in \; S_k) = Q_{n-k+1}$$ **M1**

(by similar reasoning as in part (ii))

$$P(T_{n-1} \; sits \; in \; S_{n-1} | T_1 \; sits \; in \; S_1 \; or \; S_n) = 1$$ **M1**

Therefore

$$Q_n = \frac{1}{n}\left(2 + \sum_{k=2}^{n-2} Q_{n-k+1}\right) = \frac{1}{n}\left(2 + \sum_{r=3}^{n-1} Q_{n-k+1}\right)$$ **A1**

Base case:
If $n = 3$, then T_2 sits in seat S_2 in any case where T_1 does not sit in seat S_2 **B1**
Suppose $Q_k = \frac{2}{3}$ for some $3 \le k < n$: **M1**
$Q_n = \frac{1}{n}\left(2 + (n-3) \times \frac{2}{3}\right) = \frac{2}{3}$ **A1**
Therefore, by induction $Q_n = \frac{2}{3}$ for $n \ge 3$ **E1**

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(i) Player A wins the match on game n with probability $p_A(1-p_A-p_B)^{n-1}$ **B1**

The probability that A wins the match is the sum to infinity of a geometric series with **M1**

$a = p_A, r = 1 - p_A - p_B$ **M1**

$$\frac{p_A}{p_A + p_B} \quad \boldsymbol{AG}$$ **M1** **A1**

(ii) The difference between the number of games won by the two players is initially 0 and either increases or decreases by 1 after each game. **E1**

Therefore, it can only be an even number (and so the match can only be won) after an even number of games. **E1**

Considering pairs of turns at a time **M1**

The game is equivalent to that in part (i), with $p_A = p^2$ and $p_B = q^2$, **M1**

and $0 < p_A + p_B < 1$ **M1**

so the probability that A wins the match is

$$\frac{p^2}{p^2 + q^2} \quad \boldsymbol{AG}$$ **A1**

(iii) Version 1:

The player has to win round 1 for the game to continue (with probability p). **M1**

Following that the game is equivalent to that in part (ii), so the probability that the player wins overall is **M1**

$$\frac{p^3}{p^2 + q^2}$$ **A1**

Version 2:

The only way for the player to win is by winning two rounds in a row, so with probability **M1**

$$p^2$$ **A1**

$$p^2 - \frac{p^3}{p^2 + q^2} = \frac{p^4 + p^2q^2 - p^3}{p^2 + q^2}$$ **M1**

$$= \frac{p^4 + p^2 - 2p^3 + p^4 - p^3}{p^2 + q^2}$$

$$= \frac{2p^4 - 3p^3 + p^2}{p^2 + q^2}$$

$$\frac{p^2(2p-1)(p-1)}{p^2 + q^2}$$ **M1**

If $1 > p > \frac{1}{2}$, $\frac{p^2(2p-1)(p-1)}{p^2+q^2} < 0$, so the player is more likely to win in version 1 (the cautious version) **AG** **E1**

If $0 < p < \frac{1}{2}$, $\frac{p^2(2p-1)(p-1)}{p^2+q^2} > 0$, so the player is more likely to win in version 2 (the bold version) **AG** **E1**

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