Cambridge Assessment Admissions Testing

Sixth Term Examination Paper [STEP]

Mathematics 2 [9470]

2020

Examiner's Report

Worked Solutions

This document was initially designed for print and as such does not reach accessibility standard WCAG 2.1 in a number of ways including missing text alternatives and missing document structure.

If you need this document in a different format please email admissionstesting@cambridgeassessment.org.uk telling us your name, email address and requirements and we will respond within 15 working days.

STEP MATHEMATICS 2

2020

Examiner's Report

STEP 2: Examiner's Report

There were just over 800 entries for this paper, and good solutions were seen to all of the questions. Candidates should be aware of the need to provide clear explanations of their reasoning throughout the paper (and particularly in questions where the result to be shown is given in the question). Short explanatory comments at key points in solutions can be very helpful in this regard, as can clearly drawn diagrams of the situation described in the question. The paper included a few questions where a statement of the form “A if and only if B” needed to be proven – candidates should be aware of the meaning of such statements and make sure that both directions of the implication are covered clearly.

In general, candidates who performed better on the questions in this paper recognised the relationships between the different parts of each question and were able to adapt methods used in earlier parts when working on the later sections of the question.

Question 1

This was the most popular question on the paper, and also the one on which candidates performed the best.

In general, candidates were confident in applying the substitution given in part (i) and completed the integration correctly, although there were a number of solutions in which careless errors were seen.

Successful completion of the remaining two parts required candidates to understand the reason why the substitution suggested in part (i) was helpful, and so candidates who continued to apply the same substitution from part (i) to later parts were often unable to make useful progress, particularly on part (iii).

In part (ii) many candidates took the approach of making a sequence of two substitutions to reach the answer. Attempts involving just one substitution were more likely to include errors, although a number of these were also completed successfully. In some cases candidates recognised that the integral was going to be reduced to a form similar to that in part (i), but then did not obtain exactly the correct function to be integrated.

In part (iii) candidates who were continuing with the same substitution as in part (i) often spent a lot of time rearranging the function to be integrated without success. Some attempts to apply partial fractions were seen in this section despite the fact that the factors were square roots of linear expressions. Many of those who were able to adapt the substitution from part (i) did recognise the form of the new function to be integrated and often selected an appropriate substitution.

Question 2

This was a popular question and candidates in general achieved good marks. Most candidates approached part (i) with reasonable confidence and followed the question's intended path of separating variables and integrating directly. Of these, the majority failed to consider the integral of $\frac{1}{x}$ as $\ln|x|$. This was entirely understandable for work on C1 as the question specifies that it lies entirely in the first quadrant. While $\ln(xy)$ was okay to deal with in the case where both x and y were negative, it would be good for candidates to be clear about the way in which the modulus function is being dealt with here.

Around 1 in 5 candidates took the alternate route of differentiating the given answer to show that it fitted the differential equation in the question. Unfortunately, not only is this much more demanding work, but almost all such attempts failed to realise the need to check the initial condition $x = y = 1$ as part of the solution. Candidates should be clear about the distinction between "show" and "verify" in such questions.

In part (ii), the sketching of two fairly straightforward functions caused unexpected difficulties when it came to putting them together suitably on the same diagram; it was important to show that the two curves intersect twice and many sketches failed to have them crossing more than once, if at all. This made the subsequent reasoning and sketch of C1 very awkward. Even for those who got this far entirely successfully, it was a common problem to find the sketch of C1 drawn without the helpful guidance supplied within the question and any results arising from correct working to date. In particular, it was important for candidates to demonstrate the symmetry in the line $y = x$ and the restrictions provided by the lines $x + y = 2$ and $x + y = 4$, all of which really should have appeared on the diagram.

Many attempts petered out by the time of part (iii), and very few candidates attempted to consider a graphical method (directly analogous to the method promoted in part (ii)) to show that the curve of C2 was constrained by the line $x + y = -2$. By the time they came to draw this second solution to the original differential equation, many candidates had forgotten either or both of the given bits of information; namely, that there was symmetry in $y = x$ and that C2 existed only in the 3rd-quadrant. Many candidates just assumed that C2 was the reflection of C1.

Candidates should be advised not to attempt to sketch curves by plotting points, as was seen in a number of cases. Instead, the information established in earlier parts of the question should be used to ensure that the key points are marked and that the shape is correct.

Question 3

This was the second least attempted of the pure questions. Relatively few candidates made a complete attempt at all of the parts and only 4 achieved full marks for the question.

The question consisted of a succession of given results which were to be established. Thus, candidates needed to be more aware of the importance of providing careful and thorough explanations and justifications for each step that they took along the way. Many marks were lost as a result of carelessness in providing all of the necessary details.

A significant number of candidates thought that the implication in (i) showed that the sequence was either increasing or decreasing and so got little or no credit. Establishing the given relations in (i) was generally done quite well, with candidates demonstrating a considerable range of algebraic skills in their working. But then a lot of candidates failed to show that the sequence was positive, which undermined their attempts to deduce that the sequence was unimodal.

Many candidates used an induction proof for the first proof in part (ii) despite the fact that a more direct approach was possible and considerably simpler. The "asked-for" induction proof was usually handled well, though establishing the baseline case was often flawed; many either overlooked the need to establish both of the cases n = 1 and n = 2 or, when giving a one-step induction proof with the help of the previously-established result, chose an incorrect baseline case.

The final part of the question was often avoided, though full attempts often gained full credit. Again, the usual oversight was to fail to establish positivity. Many of those who produced only a faltering solution here overlooked the need to compare successive terms, usually merely working with an expression for u_r only, often with the use of differentiation - attempts along such lines invariably lost all of the final 7 marks allocated, primarily because the required result is based on discrete values of r while calculus works with continuous values.

Question 4

Most candidates could justify the triangle inequality in part (i) (as well as arguing that the shortest distance is the straight line, there were successful uses of the cosine rule or $c = a\cos B + b\cos A$), but were less confident in proving the converse in part (ii). Successful approaches were to consider two circular loci for the SSS construction, or to fix two sides and vary the angle between them; in both cases care was needed to ensure all three inequalities were actually used, for example checking that neither circle can contain the other, which was often omitted. There was one elegant solution using three pairwise tangent circles.

A reasonable number of candidates obtained correct answers of “always”, “sometimes” (by examples) and “never” for part (iii) A, B and C respectively, although it was surprisingly common to forget that a, b, c must be the sides of a triangle. However, B caused some confusion as many candidates spent some time trying to prove that the new lengths did satisfy the triangle inequality.

Parts (iii) D and (iv) were found much harder and many candidates did not attempt them. A reasonable number of candidates were able to make some progress, but there were few complete solutions to either of these parts. Common errors for (iii) D were showing that the sum of the three inequalities is a true statement, or attempting to prove a positive result by examples. Most substantial attempts at (iv) used a different approach of fixing the order of a, b, c and reducing the problem to proving one of the three inequalities.

Question 5

This was a popular question and many of the solutions made good progress on the early parts of the question.

The majority of candidates gained full marks for part (i), but some candidates did not mention that $x - d(x) \geq 0$.

There was a wide range of marks achieved on part (ii). The proof that $x - 44d(x)$ is a multiple of 9 if and only if x is a multiple of 9 was completed well by those who managed to prove the result, but the majority of other attempts seen did not score any marks. In a small number of cases only the "if" direction was proved. Those who were unable to prove the first result in this part were often able to continue and find the required bounds on x however. Candidates who had completed both of these parts generally managed to find the correct answer $x = 792$, but did not necessarily fully justify that it was the only one.

Most candidates scored low marks on part (iii). It was very common to see an insufficient proof that $9|x$. Without guidance from the question as to how to find bounds on x, students produced a wide range of approaches; better bounds were needed if the student only used $107|x$, but the simple bound $d\big(d(x)\big) \leq d(x)$ together with divisibility by 963 was sufficient.

Question 6

This was the second most popular question on the paper. In general, candidates need to be careful when proving statements of the form “A if and only if B” and should be aware that in some cases it may not be possible to prove both directions in one go. Candidates should also be aware that, in some cases, the algebra is not sufficient on its own to demonstrate the reasoning and explanations of the steps are often helpful.

Part (i) of the question was generally completed well. In part (ii) many largely successful attempts were seen, but few candidates picked up all of the marks for this section. The main errors arose from not adequately considering cases and so dividing by 0, and from not noticing that $a^2 = d^2 = 1$ could result in $a = -d = \pm 1$. The most successful attempts in this part were the ones that separated the two directions of the implications. Many candidates misused the condition $M \neq \pm I$ in trying to prove the implication in one direction or did not check this condition when proving the implication in the opposite direction.

Few attempts at part (iii) were seen and a common mistake was to do the component-wise algebra to find M^4 instead of using the results from previous parts. In general, those who had understood the previous parts and attempted part (iv) were able to solve the final part of the question.

Question 7

This was the least popular of the pure questions and also the one on which marks were lowest on average.

Many candidates were able to show the first result, that $|w-1|$ is independent of t. However, candidates often did not explain well enough the connection between the form of z and the line $Re(z)=3$.

The next part of part (i) required noting that the centre lies on the real axis and working out $|w-c|$. Some candidates guessed the value of c. Common mistakes here included guessing $c=1, p-2$, or failing to note conditions in which $|w-c|$ is independent of t. In many solutions the absolute value sign on the radius was forgotten.

Part (ii) was similar to the previous part but required noting that the centre lies on the imaginary axis and working out $|w-ci|$. In both parts a common attempt was the guess the centre to be at a point $z=a+bi$, few candidates were successful using this method. Again, absolute value signs on the radius were regularly forgotten.

Another successful method employed by candidates in all parts of the question was to use the substitution $t=\tan\frac{\theta}{2}$, $t=(p-2)\tan\frac{\theta}{2}$, $t=q\tan\frac{\theta}{2}+2$ and using various trig identities to achieve the centre and radius. A few students also expressed t in terms of Re(w) and Im(w) and used that to obtain the equation of a circle in $\mathbb{R}^2$.

Question 8

For the first part candidates were asked to sketch a curve $y = F(x)$ based on some information about the function $f(x)$. A not insignificant number of candidates instead sketched $y = f(x)$ but those who sketched the correct curve often earned most of the marks. When justifying the given form of $F(x)$ some good explanations were provided, but in many cases the repeated roots at $x = 0$ and $x = c$ were not explained. The final section of part (i) was generally completed well by those who reached it.

The next part was found to be more difficult with many candidates mistakenly using the local maximum of $F(x)$ at $x = b$ to justify the first inequality instead of the local minimum at $x = a$. It was common to see justification such as $|F(x)|<F(b)$ without showing first that $F(b) = -F(a)$. Candidates who spotted the connection with part (i) and substituted $x = b$ into their expression for $F(x) + F(c - x)$ were usually able to show $c - b = a$ or $c > 2h$. For the last section of part (ii), those who realised the connection with the first paragraph had no issues.

Candidates who reached the final part of the question were often able to obtain expression for $f(x)$ and most realised that they needed to calculate $f''(x)$ in order to find the inflection points. However, the final mark for spotting that the roots of $f''(x) = 0$ are necessarily roots of $f(x) = 0$ without explicitly calculating them (and thereby wasting time) eluded the majority of candidates who reached this part.

Question 9

This was a question that was found to be difficult.

In general, this question was not attempted well, with very few candidates progressing past the first section. Most candidates managed to pick up all the marks in the initial section of the question. However, a significant minority of students could not set up the problem correctly or knew a lot of linear acceleration (suvat) equations but could not apply them correctly (for example mistaking displacement for position) and received zero marks. Some candidates eliminated t in favour of x and could not progress to the last calculation.

Around half the candidates picked up full marks for part (i). However, many candidates tried to reason with words – almost always unsuccessfully, often believing that the particle projected from point A could not pass through the line AB.

Most of the candidates received zero marks for part (ii), failing to realise that the result follows from the height of the particle at the time of collision being non-negative. Some tried to use conservation of momentum or energy, or the equation $\mathrm{v}^2 = \mathrm{u}^2 + 2\mathrm{as}$ due to the answer being suggestive of velocity squared. Candidates who were able to progress well on this part generally achieved all of the marks.

Very few candidates progressed to part (iii) and the attempts were often poor. Candidates who did know how to proceed to the result often did not justify bounds they used to obtain inequalities.

A significant number of candidates attempted the final part of the question having omitted earlier parts. In many cases candidates did not fully appreciate the requirements when asked to show a statement of the form “A if and only if B”.

Question 10

This was the least popular question on the paper and was found to be quite difficult by those who did attempt it.

It is very useful in questions of this type to produce a good sketch of the situation described and, unfortunately, many attempts did not do this. The candidates who made good progress did tend to have good diagrams.

When resolving forces, it is useful to consider the different directions that could be chosen. In the case of this question, many candidates chose to resolve horizontally and vertically and as a result produced more complicated equations to deal with. While in most cases they were able to simplify the equations that they obtained, this method did result in more work than was necessary. When dividing equations and inequalities by expressions, candidates should be careful to consider whether the expressions are known to be non-zero (and in the case of inequalities that it is known whether it is positive or negative).

In the second part of the question, candidates struggled with the more complicated algebraic expressions that had to be manipulated and many candidates gave up before reaching the end of the question.

Question 11

This was the more popular of the probability questions and many good attempts were seen, although the majority were incomplete and only attempted the first one or two parts. Some candidates made errors when dealing with the conditional probabilities, often thinking for example that $P(A\ wins)$ could be obtained by adding $P(A\ wins|H\ first)$ and $P(A\ wins|T\ first)$. In general, those that were able to work confidently with the conditional probabilities were able to perform very well on this question.

In the first part, a number of candidates failed to consider that games could begin with either heads or tails when showing that the probability that the game never ends is 0. Additionally, some candidates assumed that $p = q = \frac{1}{2}$ for the first part of the question, although they often did then use correct expressions in terms of p and q in the later parts of the question.

In part (ii) some candidates tried to find a way to enumerate all possible sequences for any total number of flips, but this approach almost always resulted in some of the possible cases being omitted.

Many candidates failed to spot that the solution to the third part could be found by an analogous method to that used in part (ii) and so in many cases no attempt was made at this final part.

Question 12

This question had the second smallest number of attempts on the paper. Many of these successfully completed the first part of the question, but then made little progress in the later sections.

Part (i) was generally well done, although some candidates did not appreciate that $\sum \varepsilon_i = 0$ when determining whether a fair or biased die is more likely to show the same score on two successive rolls. Where candidates were able to see the connection between the first and second parts of the question, the required result was generally proven clearly.

Part (iii) could be approached in a similar way to part (i), but many of the candidates who reached this point failed to deal with the more complicated terms that arise from the expansion. Correct solutions to this part were generally very well set out.

STEP MATHEMATICS 2

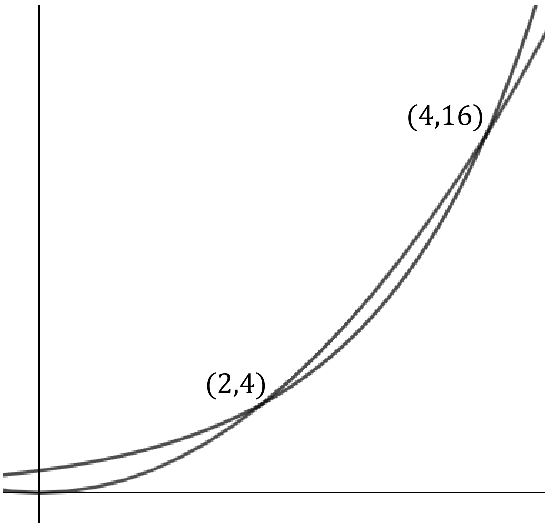
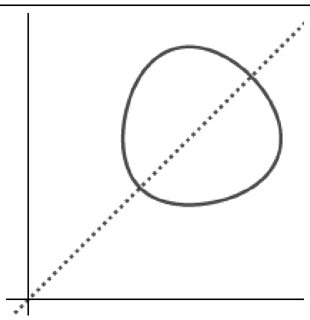
2020

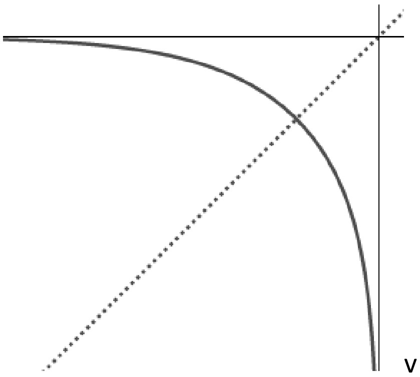
Worked Solutions

STEP 2: BRIEF SOLUTIONS

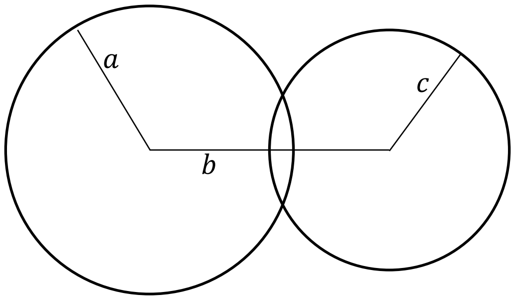
	Only penalise missing +c once in parts (i) and (ii)
1(i)	$\int \frac{1}{x^{\frac{3}{2}}(x-1)^{\frac{1}{2}}}\,\mathrm{d}x = \int \frac{(1-u)^2}{u^{\frac{1}{2}}}\frac{1}{(1-u)^2}\,\mathrm{d}u$ Must include attempt at $\frac{du}{dx}$ (or $\frac{dx}{du}$)
	$= 2u^{\frac{1}{2}}$
	$= 2\left(\frac{x-1}{x}\right)^{\frac{1}{2}} + c$
1(ii)	Let $x - 2 = s$
	Then $\int \frac{1}{(x-2)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}}\,\mathrm{d}x = \int \frac{1}{s^{\frac{3}{2}}(s+3)^{\frac{1}{2}}}\,\mathrm{d}s$
	Let $s = \frac{3}{u-1}$
	$\int \frac{1}{s^{\frac{3}{2}}(s+3)^{\frac{1}{2}}}\,\mathrm{d}s = \int \frac{(u-1)^2}{3^2u^{\frac{1}{2}}}\frac{-3}{(u-1)^2}\,\mathrm{d}u = -\tfrac{2}{3}u^{\frac{1}{2}}$
	$= -\tfrac{2}{3}\left(\frac{s+3}{s}\right)^{\frac{1}{2}} = -\tfrac{2}{3}\left(\frac{x+1}{x-2}\right)^{\frac{1}{2}} + c$
1(iii)	Let $x = \frac{1+u}{u}$ Allow substitution leading to two algebraic factors in the denominator.
	$\int_2^{\infty} \frac{1}{(x-1)(x-2)^{\frac{1}{2}}(3x-2)^{\frac{1}{2}}}\,\mathrm{d}x = \int_1^0 \frac{u^2}{(1-u)^{\frac{1}{2}}(3+u)^{\frac{1}{2}}}\cdot\left(\frac{-1}{u^2}\right)\mathrm{d}u$ If done through a sequence of substitutions: a further substitution leading to a square root of a quadratic as the denominator.
	$= \int_0^1 \frac{1}{\left(3-2u-u^2\right)^{\frac{1}{2}}}\,\mathrm{d}u$

	$= \int_0^1 \frac{1}{\left(4-(1+u)^2\right)^{\frac{1}{2}}} \, \mathrm{d}u$
	$= \left[\arcsin\left(\frac{1+u}{2}\right)\right]_0^1$
	$= \frac{1}{2}\pi - \frac{1}{6}\pi = \frac{1}{3}\pi$

2(i)	$\dfrac{1-ky}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{kx-1}{x}$
	$\ln\lvert y\rvert-ky=kx-\ln\lvert x\rvert+c$
	Hence, $\ln\lvert xy\rvert=k\left(x+y\right)+c$
	$xy=\frac{1}{4}\left[\left(x+y\right)^2-\left(x-y\right)^2\right]=Ae^{k(x+y)}$
	C_1 is $\left(x-y\right)^2=\left(x+y\right)^2-2^{x+y}$
	C_2 is $\left(x-y\right)^2=\left(x+y\right)^2-2^{x+y+4}$
	In both cases, the equation is invariant under $(x,y)\mapsto(y,x)$, so symmetrical in $y=x$.
2(ii)	
	Graphs: Correct shapes of curves
	Graphs: Intersections at (2,4) and (4,16)
	$(x-y)^2\geq 0$, so $(x+y)^2>2^{x+y}$
	Therefore, $(x+y)$ must lie between 2 and 4
	
	Graph: Symmetry about $y=x$
	Graph: Closed curve lying between $x+y=3\pm 1$
	Graph: Passes through (1,1) and (2,2)

2(iii)	Sketches of $y = x^2$ and $y = 2^{x+4}$ $x^2 > 2^{x+4}$ only when $x < -2$.
	
	Graph: Symmetry about $y = x$
	Graph: Passes through (-1,-1)
	Graph: $y \to 0$ as $x \to \infty$, $y \to -\infty$ as $x \to 0$

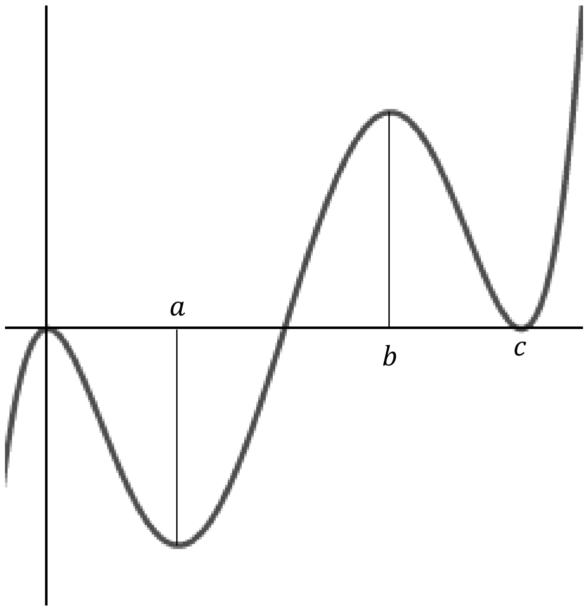
3(i)	Suppose, $\exists k: 2 \le k \le n-1$ such that $u_{k-1} \ge u_k$, but $u_k < u_{k+1}$
	Since all of the terms are positive, these imply that $u_k^2 < u_{k-1}u_{k+1}$, so the sequence does not have property *L*.
	Therefore, if the sequence has property *L*, once a value k has been reached such that $u_{k-1} \ge u_k$, it must be the case that all subsequent terms also have that property (which is the given definition of unimodality).
3(ii)	$u_r - \alpha u_{r-1} = \alpha\left(u_{r-1} - \alpha u_{r-2}\right)$, so $u_r - \alpha u_{r-1} = \alpha^{r-2}\left(u_2 - \alpha u_1\right)$
	$u_r^2 - u_{r-1}u_{r+1} = u_r^2 - u_{r-1}\left(2\alpha u_r - \alpha^2 u_{r-1}\right) = \left(u_r - \alpha u_{r-1}\right)^2$ for $r \geqslant 2$
	The first identity shows that $u_r > 0$ for all *r* if $u_2 > au_1 > 0$.
	Since the right hand side of the second identity is always non-negative, the sequence has property *L*, and is hence unimodal.
3(iii)	$u_1 = (2-1)\alpha^{1-1} + 2(1-1)\alpha^{1-2} = 1$, which is correct. $u_2 = (2-2)\alpha^{2-1} + 2(2-1)\alpha^{2-2} = 2$, which is correct.
	Suppose that: $u_{k-2} = (4-k)\alpha^{k-3} + 2(k-3)\alpha^{k-4}$, and $u_{k-1} = (3-k)\alpha^{k-2} + 2(k-2)\alpha^{k-3}$.
	$u_k = 2\alpha\left((3-k)\alpha^{k-2} + 2(k-2)\alpha^{k-3}\right) - \alpha^2((4-k)\alpha^{k-3} + 2(k-3)\alpha^{k-4})$ $= \alpha^{k-1}(6-2k-4+k) + \alpha^{k-2}(4k-8-2k+6)$ $= \alpha^{k-1}(2-k) + 2\alpha^{k-2}(k-1)$ which is the correct expression for u_k
	Hence, by induction $\mathrm{u_r} = (2-\mathrm{r})\alpha^{\mathrm{r}-1} + 2(\mathrm{r}-1)\alpha^{\mathrm{r}-2}$
	$u_r - u_{r+1} = \left((2-r)\alpha^{r-1} + 2(r-1)\alpha^{r-2}\right) - \left((1-r)\alpha^r + 2r\alpha^{r-1}\right)$
	$= \alpha^{r-2}\left(2(r-1) + (2-3r)\alpha + (r-1)\alpha^2\right)$
	$= \dfrac{\alpha^{r-2}}{N^2}\left(2N^2(r-1) + (2-3r)N(N-1) + (r-1)(N-1)^2\right)$ $= \dfrac{\alpha^{r-2}}{N^2}\left((r-1) + rN - N^2\right)$
	when $r = N$, $u_N - u_{N+1} = \dfrac{\alpha^{r-2}(N-1)}{N^2} > 0$
	when $r = N-1$, $u_{N-1} - u_N = \dfrac{-2\alpha^{r-2}}{N^2} < 0$
	so u_r is largest when $r = N$

4(i)	The straight line distance between two points must be less than the length of any other rectilinear path between the points.
4(ii)	
	Diagram showing two circles and straight line joining their centres. Length of line and radii of circles are a, b and c in some order.
	Either statement that the straight line is the longest of the lengths, or explanation that one circle cannot be contained inside the other.
	Explanation that the circles must meet.
4(iii)	**(A)** If $a+b>c$ then $(a+1)+(b+1)>c+2>c+1$ *et cycl.*, so $a+1, b+1, c+1$ can always form the sides of a triangle.
	(B) If $a=b=c=1$ we have 1, 1, 1 which can form the sides of a triangle.
	If $a=1, b=c=2$ we have $\frac{1}{2}$, 1, 2 which cannot form the sides of a triangle.
	Therefore, $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ can sometimes, but not always form the sides of a triangle.
	(C) If $p \geq q \geq r$ then $\lvert p-q\rvert + \lvert q-r\rvert = p-q+q-r = p-q = \lvert p-r\rvert$
	So two of $\lvert p-q\rvert$, $\lvert q-r\rvert$, $\lvert p-r\rvert$ will always sum to the third, so they never form the sides of a triangle.
	(D) If $a+b>c$ then $a^2+bc+b^2+ca = a^2+b^2-2ab+c(a+b)+2ab$
	$=(a-b)^2+c(a+b)+2ab>c^2+ab$ *et cycl.* so a^2+bc, b^2+ca, c^2+ab can always form the sides of a triangle.
4(iv)	Since $a+b>a \text{ and } b$, $\frac{f(a)}{a}>\frac{f(a+b)}{a+b}$ and $\frac{f(b)}{b}>\frac{f(a+b)}{a+b}$
	Since $c<a+b$, $f(c)<f(a+b)$
	Thus $f(a)+f(b)>\frac{af(a)}{a+b}+\frac{bf(b)}{a+b}=f(a+b)>f(c)$ *et cycl.* So $f(a)$, $f(b)$ and $f(c)$ can form the sides of a triangle.

5(i)	$x - q(x) = \sum_{r=0}^{n-1} a_r \times 10^r - \sum_{r=0}^{n-1} a_r = \sum_{r=0}^{n-1} a_r \times (10^r - 1)$
	$10^r \geq 1 \quad \forall r$, so $x - q(x)$ is non-negative
	$9 \mid (10^r - 1) \quad \forall r$
5(ii)	$x - 44q(x) = 44\big(x - q(x)\big) = 43x$
	So it is a multiple of 9 iff $43x$ is.
	$(43,9) = 1$, so $x - 44q(x)$ is a multiple of 9 iff x is
	If x has n digits, $q(x) \leq 9n$
	Since $x = 44q(x)$, $x \leq 396n$. Any n digit number must be at least 10^{n-1}.
	These inequalities cannot be simultaneously true for $n \geq 5$ $(396 \times 5 < 10^4)$. Therefore $n \leq 4$.
	Since $x - 44q(x) = 0$, which is a multiple of 9, x is a multiple of 9.
	$q(x)$ is an integer and $x = 44q(x)$, so x is a multiple of 44. Since $(9,44) = 1$, x must be a multiple of $44 \times 9 = 396$.
	So $x = 396k$ and therefore (by the result above) $k \leq 4$.
	Checking: Only $k = 2$ works.
5(iii)	$x - 107q\big(q(x)\big) = 0 = 107\big(x - q(x)\big) + 107\left(q(x) - q\big(q(x)\big)\right) - 106x$
	$(x - q(x))$ and $(q(x) - q\big(q(x)\big))$ are both divisible by 9 (by part (i)) and so x is divisible by 9
	$x = 107q(q(x))$ and so is divisible by 107, and so is divisible by 963. So $x = 963k$ for some k.
	If x has n digits, then $q(x) \leq 9n$. By (i), $q\big(q(x)\big) \leq q(x) \leq 9n$. So $x \leq 963n$ and $x \geq 10^{n-1}$ which implies that $n \leq 4$ and so $k \leq 4$
	Checking: Only $k = 1$ works.

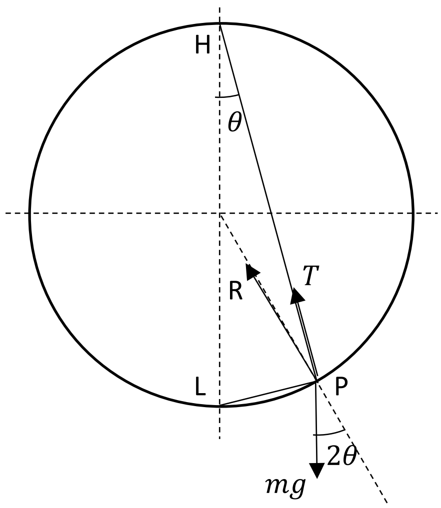
6(i)	Let $\mathbf{M}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then $\mathbf{M}^2=\begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix}$
	so Tr($\mathbf{M}^2$) = $a^2+d^2+2bc=(a+d)^2-2(ad-bc)$
6(ii)	Let $\mathbf{M}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then $\mathbf{M}^2=\begin{pmatrix} a\tau-\delta & b\tau \\ c\tau & d\tau-\delta \end{pmatrix}$, where τ = Tr(**M**) and δ = Det(**M**).
	Thus $\mathbf{M}^2=\pm\mathbf{I} \Leftrightarrow \tau=0$ and $\delta=\mp 1$ or $b=c=0$ and $a^2=d^2=\pm 1$
	If $b=c=0$ and $a=d=\pm 1$, then $\mathbf{M}=\pm\mathbf{I}$
	If $b=c=0$ and $a=-d=\pm 1$, then $\tau=0$ and $\delta=-1$
	Thus $\mathbf{M}^2=+\mathbf{I} \Leftrightarrow \tau=0$ and $\delta=-1$.
	Thus $\mathbf{M}^2=-\mathbf{I} \Leftrightarrow \tau=0$ and $\delta=+1$.
6(iii)	Part **(ii)** implies Det($\mathbf{M}^2$) = – 1, if $\mathbf{M}^4=\mathbf{I}$, but $\mathbf{M}^2\neq\pm\mathbf{I}$.
	However, Det($\mathbf{M}^2$) = Det($\mathbf{M}$)^2, so this is impossible.
	Clearly $\mathbf{M}^2=\pm\mathbf{I}\Rightarrow\mathbf{M}^4=\mathbf{I}$
	Part **(ii)** implies that $\mathbf{M}^4=-\mathbf{I}\Leftrightarrow$ Tr($\mathbf{M}^2$) = 0 and Det($\mathbf{M}^2$) = 1
	so from **(i)** $\Leftrightarrow$ Tr($\mathbf{M}$)^2 = 2Det($\mathbf{M}$) and Det($\mathbf{M}$) = ± 1
	so $\Leftrightarrow$ Tr($\mathbf{M}$) = $\pm\sqrt{2}$ and Det($\mathbf{M}$) = 1.
	Any example, for instance a matrix satisfying the conditions for any of $\mathbf{M}^2=\mathbf{I}$, $\mathbf{M}^2=-\mathbf{I}$, $\mathbf{M}^4=-\mathbf{I}$, which is not a rotation or reflection.

7(i)	$\lvert w-1\rvert^2 = \left\lvert\frac{1-ti}{1+ti}\right\rvert^2 = \frac{(1-ti)(1+ti)}{(1+ti)(1-ti)} = 1$, which is independent of t.
	Points on the line $Re(z) = 3$ have the form $z = 3 + ti$ and the points satisfying $\lvert w-1\rvert = 1$ lie on a circle with centre 1.
	If $z = p + ti$, then $\lvert w-c\rvert^2 = \left\lvert\frac{2-(p-2)c-cti}{(p-2)+ti}\right\rvert^2 = \frac{(2-(p-2)c)^2+c^2t^2}{(p-2)^2+t^2}$
	which is independent of t when $(2-(p-2)c)^2 = c^2(p-2)^2$
	which is when $c = \frac{1}{p-2}$. Thus the circle has centre at $\frac{1}{p-2}$ and radius $\frac{1}{\lvert p-2\rvert}$.
	$w = \frac{2}{(p-2)+ti} = \frac{2(p-2)-2ti}{(p-2)^2+t^2}$,
	so Im(w) > 0 when $t < 0$; that is, for those z on V with negative imaginary part.
7(ii)	If $z = t + qi$ then $\lvert w-ci\rvert^2 = \left\lvert\frac{2+cq-(t-2)ci}{(t-2)+qi}\right\rvert^2 = \frac{c^2(t-2)^2+(cq+2)^2}{(t-2)^2+q^2}$
	which is independent of t when $(cq+2)^2 = c^2q^2$
	which is when $c = -\frac{1}{q}$ so the circle has centre $-\frac{1}{q}i$ **A1** and radius $\sqrt{c^2} = \frac{1}{\lvert q\rvert}$ **A1**.
	$w = \frac{2}{(t-2)+qi} = \frac{2(t-2)-2qi}{(t-2)^2+q^2}$,
	so Re(w) > 0 when $t > 2$; that is, for those z on H with real part greater than 2.

8(i)	
	Graph: Zeroes at $x = 0, c$ and one other point (h: label not required) in (a, b).
	Graph: Turning points at $x = 0, a, b, c$.
	Graph: Quintic shape with curve below axis in $(0, h)$ and above axis in (h, c)
	The area conditions give $F(0) = F(c) = 0$. $F'(x) = f(x)$, so $F'(0) = F'(a) = F'(b) = F'(c) = 0$
	Since f is a quartic and the coefficient of x^4 is 1, F must be a quintic and the coefficient of x^5 is $\frac{1}{5}$. $F(0) = F'(0) = 0$ and $F(c) = F'(c) = 0$, so F must have double roots at $x = 0$ and c. So $F(x)$ must have the given form. [Explanation must be clear that the double roots are deduced from the fact that $\mathrm{F(x) = F'(x) = 0}$ at those points.]
	$F(x) + F(c-x) = \frac{1}{5}x^2(x-c)^2[(x-h) + (c-x-h)]$ $= \frac{1}{5}x^2(c-x)^2(c-2h)$
8(ii)	Let A be the (positive) area enclosed by the curve between 0 and a. The maximum turning point of $F(x)$ occurs at $x = b$, with $F(b) = A$. The minimum turning point of $F(x)$ occurs at $x = a$, with $F(a) = -A$.
	Therefore $F(x) \geq -A$, with equality iff $x = a$. So $F(b) + F(x) \geq 0$, with equality iff $x = a$.
	$F(a) + F(x) \leq 0$, with equality iff $x = b$.
	Since $F(b) + F(c-b) = \frac{1}{5}b^2(c-b)^2(c-2h)$, either $c > 2h$, or $c = 2h$ and $c - b = a$.
	Also, $F(a) + F(c-a) = \frac{1}{5}a^2(c-a)^2(c-2h)$, so either $c < 2h$, or $c = 2h$ and $c - a = b$.
	Thus $c = a + b$ and $c = 2h$.
8(iii)	$F(x) = \frac{1}{10}x^2(x-c)^2(2x-c)$ So $f(x) = \frac{1}{5}x(x-c)(5x^2 - 5xc + c^2)$

	The roots of the quadratic factor must be a and b.
	$f(x) = \frac{1}{5}(5x^4 - 10cx^3 + 6c^2x^2 - c^3x)$ $f'(x) = \frac{1}{5}(20x^3 - 30cx^2 + 12c^2)$ $f''(x) = \frac{1}{5}(60x^2 - 60cx + 12c^2) = \frac{12}{5}(5x^2 - 5cx + c^2)$
	Therefore $f''(x) = 0$ at $x = a$ and $x = b$ and so $(a, 0)$ and $(b, 0)$ are points of inflection.

9	If the particles collide at time t: $Vt + Ut\cos\theta = d$, and $h - \frac{1}{2}gt^2 = Ut\sin\theta - \frac{1}{2}gt^2$ (or $h = Ut\sin\theta$)
	Therefore, $d\sin\theta - h\cos\theta = Vt\sin\theta + Ut\sin\theta\cos\theta - Ut\sin\theta\cos\theta$ $= \frac{Vh}{U}$
9(i)	Dividing the previous result by $d\cos\theta$ gives: $\tan\theta - \frac{h}{d} = \frac{Vh}{Ud\cos\theta} > 0$
	Since $\tan\beta = \frac{h}{d}$, $\tan\theta > \tan\beta$ and so $\theta > \beta$
9(ii)	The height of collision must be non-negative, so $Ut\sin\theta - \frac{1}{2}gt^2 \geq 0$.
	So $U\sin\theta \geq \frac{1}{2}gt = \frac{1}{2}g\frac{h}{U\sin\theta}$ or $(U\sin\theta)^2 \geq \frac{gh}{2}$ Therefore $U\sin\theta \geq \sqrt{\frac{gh}{2}}$.
9(iii)	$d\sin\theta - h\cos\theta$ can be written as $\sqrt{d^2+h^2}\sin(\theta-\beta)$
	So $d\sin\theta - h\cos\theta < \sqrt{d^2+h^2}$ (since $\theta > \beta$)
	Therefore, $\frac{Vh}{U} < \sqrt{d^2+h^2}$ or $\sin\beta = \frac{h}{\sqrt{d^2+h^2}} < \frac{U}{V}$
	The height at which the particles collide is: $h - \frac{1}{2}gt^2 = h - \frac{gh^2}{2U^2\sin^2\theta}$
	$h - \frac{gh^2}{2U^2\sin^2\theta} > \frac{1}{2}h$ iff $U^2\sin^2\theta > gh$
	The vertical velocity of the particle fired from B at the point of collision is: $U\sin\theta - gt = U\sin\theta - \frac{gh}{U\sin\theta}$
	$U\sin\theta - \frac{gh}{U\sin\theta} > 0$ iff $U^2\sin^2\theta > gh$
	Since both cases have the same condition: The particles collide at a height greater than $\frac{1}{2}h$ if and only if the particle projected from B is moving upwards at the time of collision.

10(i)	
	Diagram showing necessary forces and angles
	$T = \frac{\lambda(2a\cos\alpha - l)}{l}$
	Resolving tangentially: $T\sin\alpha - mg\sin 2\alpha = 0$
	Therefore $\sin\alpha\left(\frac{\lambda}{l}(2a\cos\alpha - l) - 2mg\cos\alpha\right) = 0$
	Since $\sin\alpha > 0$, $2a\lambda\cos\alpha - \lambda l - 2mgl\cos\alpha = 0$ $\cos\alpha = \frac{\lambda l}{2(a\lambda - mgl)}$
	$\cos\alpha < 1$, so $\lambda l < 2(a\lambda - mgl)$ Therefore $\lambda(2a - l) > 2mgl$
	Since $2a - l > 0$, $\lambda > \frac{2mgl}{2a-l}$
10(ii)	Energy: $\frac{1}{2}mv^2 - mga\cos 2\theta + \frac{\lambda}{2l}(2a\cos\theta - l)^2 = \frac{1}{2}mu^2 - mga + \frac{\lambda}{2l}(2a - l)^2$
	If the particle comes to rest when $\theta = \beta$: $-mga\,(2\cos^2\beta - 1) + \frac{\lambda}{2l}(2a\cos\beta - l)^2 = \frac{1}{2}mu^2 - mga + \frac{\lambda}{2l}(2a - l)^2$
	$a\lambda\cos^2\beta\left(\frac{2(a\lambda - mgl)}{\lambda l}\right) - 2a\lambda\cos\beta = \frac{1}{2}mu^2 - 2mga + \frac{2\lambda a^2}{l} - 2a\lambda$
	Therefore, $\cos^2\beta - 2\cos\alpha\cos\beta = \frac{mu^2}{2a\lambda}\cos\alpha + 1 - 2\cos\alpha$
	Adding $\cos^2\alpha$ to both sides: $(\cos\alpha - \cos\beta)^2 = (1 - \cos\alpha)^2 + \frac{mu^2}{2a\lambda}\cos\alpha$
	For this to occur, $\cos\beta > 0$:
	$\cos^2\alpha > (1 - \cos\alpha)^2 + \frac{mu^2}{2a\lambda}\cos\alpha$
	And so, $u^2 < \frac{2a\lambda}{m}(2 - \sec\alpha)$

11(i)	If the game has not ended after $2n$ turns, then the sequence has either been n repetitions of HT or n repetitions of TH. So $P(Game\ has\ not\ finished\ after\ 2n\ turns) = 2(pq)^n$. So the probability that the game never ends is $\lim_{n\to\infty} 2(pq)^n = 0$.
	Sequence that follows the first H will be k repetitions of TH, followed by H, where $k \geq 0$.
	So $P(A\ wins \mid first\ toss\ is\ H) = \sum_{k=0}^{\infty}(pq)^k p = \frac{p}{1-pq}$
	$P(A\ wins \cap first\ toss\ is\ H) = p \times \frac{p}{1-pq}$
	If first toss is a tail then the sequence that follows would be k repetitions of HT followed by HH.
	So $P(A\ wins \mid first\ toss\ is\ T) = \frac{p^2}{1-pq}$
	$P(A\ wins \cap first\ toss\ is\ T) = \frac{p^2 q}{1-pq}$
	Therefore $P(A\ wins) = \frac{p^2(1+q)}{1-pq}$
11(ii)	Following a first toss of H: A wins with HH or (HT followed by any sequence where A wins after first toss was T) or (T followed by any sequence where A wins after first toss was T)
	The probabilities of these cases are: p^2 $pq\ P(A \text{ wins } \mid \text{the first toss is a tail})$ $q\ P(A \text{ wins } \mid \text{the first toss is a tail})$
	Therefore: $P(\text{A wins} \mid \text{the first toss is a head}) = p^2 + (q + pq)P(\text{A wins} \mid \text{the first toss is a tail})$

	Similarly, following first toss of T: A wins with (H followed by any sequence where A wins after first toss was H) or (TH followed by any sequence where A wins after first toss was H)
	Therefore: $P(\text{A wins} \mid \text{the first toss is a tail}) = (p + pq)P(\text{A wins} \mid \text{the first toss is a head})$
	So $P(A \mid H\ first) = p^2 + (q + pq)(p + pq)P(A \mid H\ first)$ $P(A \mid H\ first) = \frac{p^2}{1-(p+pq)(q+pq)}$
	And $P(A \mid T\ first) = (p + pq)(p^2 + (q + pq)P(A \mid T\ first))$ $P(A \mid T\ first) = \frac{p^2(p+pq)}{1-(p+pq)(q+pq)}$
	So $P(A\ wins) = p \times \frac{p^2}{1-(p+pq)(q+pq)} + q \times \frac{p^2(p+pq)}{1-(p+pq)(q+pq)} = \frac{p^2(1-q^3)}{1-(1-p^2)(1-q^2)}$
11(iii)	Let W be the event that A wins the game. $P(W \mid H\ first) = p^{a-1} + (1 + p + p^2 + \cdots + p^{a-2})qP(W \mid T\ first)$
	$P(W \mid T\ first) = \left(1 + q + q^2 + \cdots + q^{b-2}\right)pP(W \mid H\ first)$
	$P(W \mid H\ first) = \frac{p^{a-1}}{1-(1-p^{a-1})(1-q^{b-1})}$
	$P(W \mid T\ first) = \frac{p^{a-1}(1-q^{b-1})}{1-(1-p^{a-1})(1-q^{b-1})}$
	Therefore: $P(W) = \frac{p^{a-1}\left(1-q^b\right)}{1-(1-p^{a-1})(1-q^{b-1})}$
	If $a = b = 2$, $P(W) = \frac{p\left(1-q^2\right)}{1-(1-p)(1-q)} = \frac{p^2(1+q)}{1-pq}$ as expected.

12(i)	For the biased die: $P(R_1 = R_2) = \sum_{i=1}^{n} \left(\frac{1}{n} + \varepsilon_i\right)^2$
	$P(R_1 = R_2) = \frac{1}{n^2}\sum_{i=1}^{n} 1 + \frac{2}{n}\sum_{i=1}^{n} \varepsilon_i + \sum_{i=1}^{n} {\varepsilon_i}^2$
	$\sum_{i=1}^{n} \varepsilon_i = 0$, so $P(R_1 = R_2) = \frac{1}{n} + \sum_{i=1}^{n} {\varepsilon_i}^2$
	For a fair die, $P(R_1 = R_2) = \frac{1}{n}$ and $\sum_{i=1}^{n} {\varepsilon_i}^2 > 0$, so it is more likely with the biased die.
(ii)	$P(R_1 > R_2) = \frac{1}{2}\big(1 - P(R_1 = R_2)\big)$
	Therefore, the value of $P(R_1 > R_2)$ if the die is possibly biased is $\leq P(R_1 > R_2)$ if the die is fair.
	Let T= $\sum_{r=1}^{n} x_r$ and, for each i, let $p_i = \frac{x_i}{T}$ Then $\sum_{i=1}^{n} p_i = 1$, so we can construct a biased n-sided die with $P(X = i) = p_i$
	$P(R_1 > R_2) = \sum_{i=2}^{n}\sum_{j=1}^{i-1} p_i p_j$
	For a fair die: $P(R_1 > R_2) = \frac{n-1}{2n}$
	Therefore $\sum_{i=2}^{n}\sum_{j=1}^{i-1} \frac{x_i x_j}{T^2} \leq \frac{n-1}{2n}$ and so $$\sum_{i=2}^{n}\sum_{j=1}^{i-1} x_i x_j \leq \frac{n-1}{2n}\left(\sum_{i=1}^{n} x_i\right)^2$$
(iii)	For the biased die: $P(R_1 = R_2 = R_3) = \sum_{i=1}^{n} \left(\frac{1}{n} + \varepsilon_i\right)^3$
	$= \sum_{i=1}^{n} \frac{1}{n^3} + \sum_{i=1}^{n} \frac{3\varepsilon_i}{n^2} + \sum_{i=1}^{n} \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^{n} {\varepsilon_i}^3$
	Therefore $$P(R_1 = R_2 = R_3\ biased) - P(R_1 = R_2 = R_3\ fair) = \sum_{i=1}^{n} \frac{3\varepsilon_i}{n^2} + \sum_{i=1}^{n} \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^{n} {\varepsilon_i}^3$$ $= \sum_{i=1}^{n} \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^{n} {\varepsilon_i}^3$ (since $\sum_{i=1}^{n} \varepsilon_i = 0$)
	$= \sum_{i=1}^{n} \frac{3\varepsilon_i^2}{n} + \varepsilon_i^3 = \sum_{i=1}^{n} \varepsilon_i^2 \left(\frac{3}{n} + \varepsilon_i\right)$
	But $\varepsilon_i \geq -\frac{1}{n}$ (since $p_i \geq 0$), so this sum must be positive.
	Therefore, $P(R_1 = R_2 = R_3)$ must be greater for the biased die.

Cambridge Assessment Admissions Testing offers a range of tests to support selection and recruitment for higher education, professional organisations and governments around the world. Underpinned by robust and rigorous research, our assessments include:

- assessments in thinking skills
- admissions tests for medicine and healthcare
- behavioural styles assessment
- subject-specific admissions tests.

We are part of a not-for-profit department of the University of Cambridge.

Cambridge Assessment
Admissions Testing
The Triangle Building
Shaftesbury Road
Cambridge
CB2 8EA
United Kingdom

Admissions tests support:

www.admissionstestingservice.org/help