STEP 2024, Paper 3, Q8 Solution (4 pages; 30/5/25)

8 **(i)** Explain why the equation $(y - x + 3)(y + x - 5) = 0$ represents a pair of straight lines with gradients 1 and -1. Show further that the equation

$$y^2 - x^2 + py + qx + r = 0$$

represents a pair of straight lines with gradients 1 and -1 if and only if $p^2 - q^2 = 4r$.

In the remainder of this question, C_1 is the curve with equation $x = y^2 + 2sy + s(s+1)$ and C_2 is the curve with equation $y = x^2$.

(ii) Explain why the coordinates of any point which lies on both of the curves C_1 and C_2 also satisfy the equation

$$y^2 + 2sy + s(s+1) - x + k(y - x^2) = 0$$

for any real number k.

Given that s is such that C_1 and C_2 intersect at four distinct points, show that choosing $k = 1$ gives an equation representing a pair of straight lines, with gradients 1 and -1, on which all four points of intersection lie.

(iii) Show that if C_1 and C_2 intersect at four distinct points, then $s < -\frac{3}{4}$.

(iv) Show that if $s < -\frac{3}{4}$, then C_1 and C_2 intersect at four distinct points.

(i) **1st Part**

If $(y-x+3)(y+x-5)=0$, then either

$y-x+3=0$ or $y+x-5=0$, and these represent straight lines with gradients 1 and -1 respectively.

2nd Part

$y^2-x^2+py+qx+r=0$ (1)

$\Leftrightarrow (y-x)(y+x)+c_2(y-x)+c_1(y+x)+c_1c_2=0$

or $(y-x+c_1)(y+x+c_2)=0$, (2)

where $p=c_1+c_2$, $q=c_1-c_2$ and $r=c_1c_2$, (3)

so that $p^2-q^2=(p-q)(p+q)=2c_2(2c_1)=4r$ (4)

If (1) represents a pair of straight lines with gradients 1 and -1, then (1) can be written as (2), with conditions (3) holding; from which (4) follows.

If $p^2-q^2=4r$, then we can write $c_1=\frac{1}{2}(p+q)$ and

$c_2=\frac{1}{2}(p-q)$, so that (3) holds, and then (1) can be written as (2); ie (1) represents a pair of straight lines with gradients 1 and -1.

This establishes the required 'if and only if' result.

(ii) **1st Part**

If the point (x, y) lies on both C_1 and C_2, then

$y^2 + 2sy + s(s+1) - x + k(y - x^2)$

$= x - x + k(x^2 - x^2) = 0$ (for any real k)

2nd Part

With $k = 1$, the given equation becomes

$y^2 + 2sy + s(s+1) - x + (y - x^2) = 0 \quad (*)$

or $y^2 - x^2 + py + qx + r = 0$,

where $p = 2s + 1, q = -1$ and $r = s(s+1)$

And $p^2 - q^2 = (p - q)(p + q) = (2s + 2)(2s) = 4r$

From the 1st Part of (ii), the 4 points of intersection satisfy (*), and from the 2nd Part of (i), $p^2 - q^2 = 4r \Rightarrow (*)$ represents a pair of straight lines with gradients 1 and -1.

(iii) From previous parts, the equation

$y^2 + 2sy + s(s+1) - x + (y - x^2) = 0$

can be written as $(y - x + c_1)(y + x + c_2) = 0$,

where $c_1 = \frac{1}{2}(p + q)$ and $c_2 = \frac{1}{2}(p - q)$,

and $p = 2s + 1, q = -1$

Also, $y = x^2$, and the equations of the two straight lines become

$x^2 - x + s = 0$ and $x^2 + x + s + 1 = 0$

For there to be two distinct solutions for both of these quadratic equations, we require:

$1 - 4s > 0$ and $1 - 4(s + 1) > 0$;

ie $s < \frac{1}{4}$ and $-3 > 4s$; $s < -\frac{3}{4}$

ie $s < -\frac{3}{4}$ QED

(iv) If $s < -\frac{3}{4}$ then the quadratic equations $x^2 - x + s = 0$ and $x^2 + x + s + 1 = 0$ each have two distinct solutions, so that the quartic $(x^2 - x + s)(x^2 + x + s + 1) = 0$ has 4 distinct solutions.

Expanding out this quartic gives

$x^4 + x^3(1 - 1) + x^2(s + 1 + s - 1) + x(-s - 1 + s) + s(s + 1)$

$= 0$, or $x^4 + 2sx^2 - x + s(s + 1) = 0$

And this is the same quartic obtained by substituting $y = x^2$ (the equation of C_2) into $x = y^2 + 2sy + s(s + 1)$ (the equation of C_1); ie C_1 and C_2 intersect at 4 distinct points.