STEP 2023, Paper 3, Q7 Solution (4 pages; 30/6/25)

7 **(i)** Let f be a continuous function defined for $0 \leqslant x \leqslant 1$. Show that

$$\int_0^1 \mathrm{f}(\sqrt{x})\,\mathrm{d}x = 2\int_0^1 x\mathrm{f}(x)\,\mathrm{d}x\,.$$

(ii) Let g be a continuous function defined for $0 \leqslant x \leqslant 1$ such that

$$\int_0^1 \left(\mathrm{g}(x)\right)^2\,\mathrm{d}x = \int_0^1 \mathrm{g}(\sqrt{x})\,\mathrm{d}x - \frac{1}{3}\,.$$

Show that $\displaystyle\int_0^1 \left(\mathrm{g}(x) - x\right)^2\,\mathrm{d}x = 0$ and explain why $\mathrm{g}(x) = x$ for $0 \leqslant x \leqslant 1$.

(iii) Let h be a continuous function defined for $0 \leqslant x \leqslant 1$ with derivative h' such that

$$\int_0^1 \left(\mathrm{h}'(x)\right)^2\,\mathrm{d}x = 2\mathrm{h}(1) - 2\int_0^1 \mathrm{h}(x)\,\mathrm{d}x - \frac{1}{3}\,.$$

Given that $\mathrm{h}(0) = 0$, find h.

(iv) Let k be a continuous function defined for $0 \leqslant x \leqslant 1$ and a be a real number, such that

$$\int_0^1 \mathrm{e}^{ax}\left(\mathrm{k}(x)\right)^2\,\mathrm{d}x = 2\int_0^1 \mathrm{k}(x)\,\mathrm{d}x + \frac{\mathrm{e}^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4}\,.$$

Show that a must be equal to 2 and find k.

(i) Writing $x = u^2$, where $u \geq 0$, $dx = 2u\,du$,

and $\int_0^1 f(\sqrt{x})dx = \int_0^1 f(u).2u\,du$ or $2\int_0^1 xf(x)\,dx$

(ii) **1st Part**

$\int_0^1 (g(x) - x)^2\,dx = \int_0^1 (g(x))^2\,dx + \int_0^1 x^2\,dx - 2\int_0^1 xg(x)dx$

$= [\int_0^1 g(\sqrt{x})\,dx - \frac{1}{3}] + \left[\frac{1}{3}x^2\right]\begin{matrix}1\\0\end{matrix} - \int_0^1 g(\sqrt{x})\,dx$, from (i)

$= -\frac{1}{3} + \left(\frac{1}{3} - 0\right) = 0$, as required.

2nd Part

Suppose that $g(x) \neq x$ for some $x \epsilon [0,1]$

As g is a continuous function, $g(x) \neq x$ for some interval of $x \epsilon [0,1]$, so that $(g(x) - x)^2 > 0$ for some such interval.

But then $\int_0^1 (g(x) - x)^2\,dx > 0$, as $(g(x) - x)^2 \geq 0$ for all x;

and this contradicts the fact that $\int_0^1 (g(x) - x)^2\,dx = 0$

Thus $g(x) = x$ for $x \epsilon [0,1]$; ie for $0 \leq x \leq 1$

(iii) Let $g(x) = h'(x)$

Then $\int_0^1 (g(x))^2\,dx = \int_0^1 (h'(x))^2\,dx$

$= 2h(1) - 2\int_0^1 h(x)dx - \frac{1}{3}$ (*)

And $\int_0^1 g(\sqrt{x})\,dx = 2\int_0^1 xg(x)\,dx$, from (i);

$= 2[xh(x)]_0^1 - 2\int_0^1 h(x)dx$, by Parts;

$= 2h(1) - 0 - 2\int_0^1 h(x)dx$,

and so, from (*) : $\int_0^1 \left(g(x)\right)^2 dx = \int_0^1 g\left(\sqrt{x}\right) dx - \frac{1}{3}$;

then, from (ii), $\int_0^1 (h'(x) - x)^2 \, dx = 0$ [noting that we haven't relied on $g = h'$ being continuous]

Then it follows that $h'(x) = x$, except possibly for isolated values of x.

And so $h(x) = \frac{1}{2}x^2 + C$, and this applies to all $0 \leq x \leq 1$, in order for h to be continuous.

Then, as $h(0) = 0, h(x) = \frac{1}{2}x^2$

(iv) Consider $\int_0^1 (e^{\frac{ax}{2}}k(x) - e^{-\frac{ax}{2}})^2 dx$

[Unfortunately, this starting point isn't that obvious. A natural approach is to try to use either (iii), or (perhaps more promisingly) (ii). However, this would only be feasible if a was already known.]

$= \int_0^1 e^{ax}(k(x))^2 dx + \int_0^1 e^{-ax} \, dx - 2\int_0^1 k(x)dx$

$= \frac{e^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4} + [-\frac{1}{a}e^{-ax}]_0^1$

$= \frac{e^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4} - \frac{1}{a}(e^{-a} - 1) = -\frac{1}{a^2} - \frac{1}{4} + \frac{1}{a}$

$= -\frac{(4+a^2-4a)}{4a^2}$

$= -\frac{(a-2)^2}{4a^2}$

Now $\int_0^1 (e^{\frac{ax}{2}}k(x) - e^{-\frac{ax}{2}})^2 dx \geq 0$, whilst $= -\frac{(a-2)^2}{4a^2} \leq 0$,

and so $\frac{(a-2)^2}{4a^2} = 0$, and therefore $a = 2$,

and $\int_0^1 (e^{\frac{ax}{2}}k(x) - e^{-\frac{ax}{2}})^2 dx = 0$,

so that $e^{\frac{ax}{2}}k(x) - e^{-\frac{ax}{2}} = 0$ for $0 \leq x \leq 1$, as in the 2nd Part of (ii);

and hence $k(x) = e^{-ax} = e^{-2x}$.