STEP 2023, Paper 3, Q5 Solution (6 pages; 19/6/25)

5 (i) Show that if
$$\frac{1}{x}+\frac{2}{y}=\frac{2}{7},$$
then $(2x-7)(y-7)=49$.

By considering the factors of 49, find all the pairs of positive integers x and y such that
$$\frac{1}{x}+\frac{2}{y}=\frac{2}{7}.$$

(ii) Let p and q be prime numbers such that
$$p^2+pq+q^2=n^2$$
where n is a positive integer. Show that
$$(p+q+n)(p+q-n)=pq$$
and hence explain why $p+q=n+1$.

Hence find the possible values of p and q.

(iii) Let p and q be positive and
$$p^3+q^3+3pq^2=n^3.$$
Show that $p+q-n<p$ and $p+q-n<q$.

Show that there are no prime numbers p and q such that $p^3+q^3+3pq^2$ is the cube of an integer.

Solution

(i) **1st Part**

$\frac{1}{x}+\frac{2}{y}=\frac{2}{7} \Rightarrow 7y+14x=2xy$

$\Rightarrow y(2x-7)-14x=0$

$\Rightarrow y(2x-7)-7(2x-7)=49$

or $(2x-7)(y-7)=49$, as required.

2nd Part

The possibilities to consider are:

$2x-7=1$ & $y-7=49$; giving $x=4, y=56$

$2x-7=-1$ & $y-7=-49$; but $y<0$

$2x-7=49$ & $y-7=1$; giving $x=28, y=8$

$2x-7=-49$ & $y-7=-1$; but $x<0$

$2x-7=7$ & $y-7=7$; giving $x=7, y=14$

$2x-7=-7$ & $y-7=-7$; but $x=0$

(ii) **1st Part**

$(p+q+n)(p+q-n)$

$=(p+q)^2-n^2$

$=p^2+q^2+2pq-n^2$

$=(p^2+pq+q^2)+pq-n^2$

$=n^2+pq-n^2$

$=pq$, as required.

2nd Part

As p & q are prime numbers, and as $p+q+n>p$

and $p+q+n>q$, the only possibility is

$p+q+n=pq$ and $p+q-n=1$,

so that $p+q=n+1$

3rd Part

[The question implies that larger values of p and q won't work.

Experimenting, we see that $p=3, q=5$ (or vice versa) works,

but no other combinations do.]

As $p+q+n=pq$ and $p+q-n=1$,

it follows that $p+q+(p+q-1)=pq$,

so that $pq-2p-2q+1=0$

or $(p-2)(q-2)=3$

This is only possible if $p=3, q=5$, or vice versa.

(iii) [As the two results to be proved reduce easily to the equivalent $q<n$ & $p<n$, the question is clearly trying to tell us something about $p+q-n$. Also, these results are clearly needed to establish the final result.

In (ii), the general idea was that the given relation involving p, q & n produced a relation between p and q, which allowed us to narrow down the possibilities for p and q, given that they were prime (and using the derived fact that $p+q=n+1$).

Notice though that we were told in (ii) that p and q were prime; which is not the case in (iii). This indicates that the prime nature of p and q is not needed for the 1st Part.

The given result that $p^3 + q^3 + 3pq^2 = n^3$ strongly suggests employing $(p+q)^3 = p^3 + 3p^2q + 3pq^2 + q^3$, to give

$(p+q)^3 - 3p^2q = n^3$,

or $(p+q)^3 - n^3 = 3p^2q$,

which gives $(p+q-n)((p+q)^2 + (p+q)n + n^2) = 3p^2q$ (*)

The question is: Are we supposed to be using (*) to derive the two results $p+q-n < p$ and $p+q-n < q$? Or are the two results to be used in conjunction with (*) to establish the final result? Hopefully the latter is the case (otherwise we don't have anything to work with for the final result). In which case, maybe our two results are easy to establish (and don't rely on the prime nature of p and q. (This turns out to be the case.)]

1st Part

As $p^3 = n^3 - q^3 - 3pq^2$, and as p and q are positive,

it follows that $p^3 < n^3$, and hence $p < n$, so that $p+q-n < q$.

And similarly $q < n$, so that $p+q-n < p$.

2nd Part

As $p^3 + q^3 + 3pq^2 = n^3$ (where n is supposed to be an integer)

$(p+q)^3 - 3p^2q = n^3$,

or $(p+q)^3 - n^3 = 3p^2q$,

which gives $(p+q-n)((p+q)^2 + (p+q)n + n^2) = 3p^2q$

If p and q are to be prime numbers, but $p+q-n < p$ and

$p + q - n < q$, then the factor $p + q - n$ on the LHS must equal 1 or 3;

so that $p + q = n + 1$ or $n + 3$

[This mirrors the result $p + q = n + 1$ in (ii).]

If $p + q = n + 1$, then [eliminating n as in (ii)]

$(p + q)^2 + (p + q)n + n^2 = 3p^2 q$,

so that $(p + q)^2 + (p + q)(p + q - 1) + (p + q - 1)^2 = 3p^2 q$

Writing $r = p + q$ for the moment,

$r^2 + r(r - 1) + (r - 1)^2 = 3p^2 q$;

$3r^2 - 3r + 1 = 3p^2 q$;

$3p^2 q - 3r^2 + 3r = 1$;

which is not possible, as the LHS is divisible by 3, whilst the RHS is not.

Whilst, if $p + q = n + 3$, then

$(p + q)^2 + (p + q)n + n^2 = p^2 q$,

so that $(p + q)^2 + (p + q)(p + q - 3) + (p + q - 3)^2 = p^2 q$

Writing $r = p + q$ for the moment,

$r^2 + r(r - 3) + (r - 3)^2 = p^2 q$;

$3r^2 - 9r + 9 = p^2 q$,

or $3(r^2 - 3r + 3) = p^2 q$

Thus 3 is a factor of $p^2 q$, and so either $p = 3$ or $q = 3$ (as p and q are prime).

But $p + q = n + 3$ then means that either $p = n$ or $q = n$, which is inconsistent with the facts that $p < n$ and $q < n$, established in the 1st Part.

Hence there are no prime numbers p and q such that

$p^3 + q^3 + 3pq^2$ is a cube of an integer.