STEP 2023, P2, Q6 - Solution (5 pages; 10/6/25)

6 The sequence F_n, for $n = 0, 1, 2, \ldots$, is defined by $F_0 = 0$, $F_1 = 1$ and by $F_{n+2} = F_{n+1} + F_n$ for $n \geqslant 0$.

Prove by induction that, for all positive integers n,

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \mathbf{Q}^n,$$

where the matrix $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

(i) By considering the matrix $\mathbf{Q}^n$, show that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ for all positive integers n.

(ii) By considering the matrix $\mathbf{Q}^{\mathbf{m+n}}$, show that $F_{m+n} = F_{m+1}F_n + F_m F_{n-1}$ for all positive integers m and n.

(iii) Show that $\mathbf{Q}^2 = \mathbf{I} + \mathbf{Q}$.

In the following parts, you may use without proof the Binomial Theorem for matrices:

$$(\mathbf{I} + \mathbf{A})^n = \sum_{k=0}^{n} \binom{n}{k} \mathbf{A}^k.$$

(a) Show that, for all positive integers n,

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k\,.$$

(b) Show that, for all positive integers n,

$$F_{3n} = \sum_{k=0}^{n} \binom{n}{k} 2^k F_k$$

and also that

$$F_{3n} = \sum_{k=0}^{n} \binom{n}{k} F_{n+k}\,.$$

(c) Show that, for all positive integers n,

$$\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} F_{n+k} = 0\,.$$

Solution

1st Part

When $n = 1$,

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} F_2 & F_1 \\ F_1 & F_0 \end{pmatrix} = \begin{pmatrix} 1+0 & 1 \\ 1 & 0 \end{pmatrix} = Q = Q^n$$

Thus the result is true for $n = 1$.

Now assume that the result is true for $n = k$,

so that $\begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix} = Q^k$

Then $Q^{k+1} = Q.Q^k = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$

$$= \begin{pmatrix} F_{k+1} + F_k & F_k + F_{k-1} \\ F_{k+1} & F_k \end{pmatrix}$$

$$= \begin{pmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{pmatrix},$$

which is the result for $n = k + 1$

Thus, if the result is true for $n = k$, then it is true for $n = k + 1$.

As the result is true for $n = 1$, it is therefore true for $n = 2, 3, \ldots$,

and hence all positive integer n, by the principle of induction.

(i) $F_{n+1}F_{n-1} - {F_n}^2 = |Q^n| = |Q|^n = (-1)^n$, as required.

(ii) As $Q^{m+n} = Q^m . Q^n$,

$$\begin{pmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{pmatrix} = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} . \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

Then, taking the top right-hand element (on the LHS),

$F_{m+n} = F_{m+1}F_n + F_mF_{n-1}$, as required (when m & n are positive integers).

(iii) **1st Part**

$$Q^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = I + Q,$$

as required.

(a) From the Binomial theorem for matrices,

$(I + Q)^n = \sum_{k=0}^{n} \binom{n}{k} Q^k$

And $(I + Q)^n = (Q^2)^n = Q^{2n} = \begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{pmatrix}$

Taking the top right-hand element again, as $Q^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$,

$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k$, as required.

(b) **1st Part**

Note that $\sum_{k=0}^{n} \binom{n}{k} 2^k Q^k = \sum_{k=0}^{n} \binom{n}{k} (2Q)^k = (I + 2Q)^n$

The top right-hand element of the LHS is $\sum_{k=0}^{n} \binom{n}{k} 2^k F_k$,

and we hope to show that $(I + 2Q)^n = Q^{3n}$

Now, $Q^3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$,

and $I + 2Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$,

so that $(I + 2Q)^n = Q^{3n}$,

the top right-hand element of which is F_{3n};

and so $F_{3n} = \sum_{k=0}^{n} \binom{n}{k} 2^k F_k$, as required.

2nd Part

Note that $\sum_{k=0}^{n} \binom{n}{k} Q^{n+k} = Q^n \sum_{k=0}^{n} \binom{n}{k} Q^k$ (*)

The top right-hand element of the LHS is $\sum_{k=0}^{n} \binom{n}{k} F_{n+k}$,

and $Q^n \sum_{k=0}^{n} \binom{n}{k} Q^k = Q^n(I + Q)^n$

Result to prove: $Q(I + Q) = (I + Q)Q$

LHS $= QI + Q^2 = Q + Q^2$

RHS $= IQ + Q^2 = Q + Q^2$ also

It follows from this that $Q^n(I + Q)^n = [Q(I + Q)]^n$,

by reordering the Qs and $(I + Q)s$.

And $Q(I + Q) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = Q^3$,

Thus the RHS of (*) is $Q^n \sum_{k=0}^{n} \binom{n}{k} Q^k = Q^n(I + Q)^n$

$= [Q(I+Q)]^n = Q^{3n}$,

and so the top right-hand element of the RHS of (*) is F_{3n},

and hence $F_{3n} = \sum_{k=0}^{n} \binom{n}{k} F_{n+k}$, as required.

(c) [After a bit of experimenting]

Consider $[-Q(I-Q)]^n = (-Q)^n \sum_{k=0}^{n} \binom{n}{k} (-Q)^k$

$= \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} Q^{n+k}$,

the top right-hand element of which is $\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} F_{n+k}$

And $-Q(I-Q) = -\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$,

so that $[-Q(I-Q)]^n = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^n = (-1)^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^n$

$= (-1)^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

the top right-hand element of which is zero;

and hence $\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} F_{n+k} = 0$, as required.