STEP 2023, P2, Q4 - Solution (4 pages; 11/6/25)

4 (i) Show that, if $(x-\sqrt{2})^2 = 3$, then $x^4 - 10x^2 + 1 = 0$.

Deduce that, if $\mathrm{f}(x) = x^4 - 10x^2 + 1$, then $\mathrm{f}(\sqrt{2}+\sqrt{3}) = 0$.

(ii) Find a polynomial g of degree 8 with integer coefficients such that $\mathrm{g}(\sqrt{2}+\sqrt{3}+\sqrt{5}) = 0$. Write your answer in a form without brackets.

(iii) Let a, b and c be the three roots of $t^3 - 3t + 1 = 0$.

Find a polynomial h of degree 6 with integer coefficients such that $\mathrm{h}(a+\sqrt{2}) = 0$, $\mathrm{h}(b+\sqrt{2}) = 0$ and $\mathrm{h}(c+\sqrt{2}) = 0$. Write your answer in a form without brackets.

(iv) Find a polynomial k with integer coefficients such that $\mathrm{k}(\sqrt[3]{2}+\sqrt[3]{3}) = 0$. Write your answer in a form without brackets.

Solution

(i) 1st Part

Method 1

$\left(x-\sqrt{2}\right)^2 = 3 \Rightarrow x^2 - 2\sqrt{2}x + 2 = 3$

$\Rightarrow x^2 - 1 = 2\sqrt{2}x$

$\Rightarrow (x^2-1)^2 = 8x^2$

or $x^4 - 10x^2 + 1 = 0$, as required.

Method 2

Write $y = x - \sqrt{2}$ [so that $y^2 = 3$], so that $x = y + \sqrt{2}$

and $x^4 - 10x^2 + 1 = \left(y+\sqrt{2}\right)^4 - 10\left(y+\sqrt{2}\right)^2 + 1$

$= \left(y+\sqrt{2}\right)^2\left[\left(y+\sqrt{2}\right)^2 - 10\right] + 1$

$= \left(y^2 + 2\sqrt{2}y + 2\right)\left(y^2 + 2\sqrt{2}y + 2 - 10\right) + 1$

$= \left(2\sqrt{2}y + 5\right)\left(2\sqrt{2}y - 5\right) + 1$, as $y^2 = 3$

$= 8y^2 - 25 + 1 = 0$, as required.

[But this method can’t be applied in (ii).]

2nd Part

If $x = \sqrt{2} + \sqrt{3}$, then $\left(x-\sqrt{2}\right)^2 = 3$,

and then from the 1st Part, $f(x) = 0$, as required.

(ii) [Unfortunately, this part is only do-able if the correct method is chosen in (i)!]

If $x = \sqrt{2} + \sqrt{3} + \sqrt{5}$, then $\left(x - \sqrt{2}\right)^2 = (\sqrt{3} + \sqrt{5})^2$,

so that $x^2 - 2\sqrt{2}x + 2 = 3 + 5 + 2\sqrt{15}$

or $x^2 - 6 = 2(\sqrt{2}x + \sqrt{15})$

$\Rightarrow (x^2 - 6)^2 = 4(\sqrt{2}x + \sqrt{15})^2$

or $x^4 + 36 - 12x^2 = 4(2x^2 + 15 + 2\sqrt{30}x)$,

or $x^4 - 20x^2 - 24 = 8\sqrt{30}x$

$\Rightarrow (x^4 - 20x^2 - 24)^2 = 64(30)x^2$

or $x^8 + 400x^4 + 576 + 2(-20x^6 - 24x^4 + 480x^2) = 1920x^2$,

or $x^8 - 40x^6 + 352x^4 - 960x^2 + 576 = 0$

(iii) Let $y = t + \sqrt{2}$.

Then the roots of $(y - \sqrt{2})^3 - 3(y - \sqrt{2}) + 1 = 0$ are $a + \sqrt{2}$, $b + \sqrt{2}$ and $c + \sqrt{2}$.

This equation can be rearrranged to give

$\left(y^3 - 3y^2\left(\sqrt{2}\right) + 3y(2) - 2\sqrt{2}\right) - 3y + 3\sqrt{2} + 1 = 0$

or $y^3 + 3y + 1 = 3y^2\left(\sqrt{2}\right) - \sqrt{2}$ (*)

The roots of (*) will then also be roots of (*) squared:

$(y^3 + 3y + 1)^2 = 2(3y^2 - 1)^2$

or $y^6 + 9y^2 + 1 + 2(3y^4 + y^3 + 3y) = 18y^4 + 2 - 12y^2$;

so that $h(y) = y^6 - 12y^4 + 2y^3 + 21y^2 + 6y - 1$

(iv) If $x = \sqrt[3]{2} + \sqrt[3]{3}$, then

$x^3 = 2 + 3(2)^{\frac{2}{3}}(3)^{\frac{1}{3}} + 3(2)^{\frac{1}{3}}(3)^{\frac{2}{3}} + 3;$

$(x^3 - 5)^3 = 27(2)(3)\left(\sqrt[3]{2} + \sqrt[3]{3}\right)^3 = 162x^3;$

$x^9 + 3x^6(-5) + 3x^3(25) + (-125) = 162x^3,$

so that $k(x) = x^9 - 15x^6 - 87x^3 - 125$