

STEP 2023, P2, Q1 - Solution (6 pages; 27/2/25)

- 1 (i) Show that making the substitution $x = \frac{1}{t}$ in the integral

$$\int_a^b \frac{1}{(1+x^2)^{\frac{3}{2}}} dx,$$

where $b > a > 0$, gives the integral

$$\int_{a^{-1}}^{b^{-1}} \frac{-t}{(1+t^2)^{\frac{3}{2}}} dt.$$

- (ii) Evaluate:

(a)
$$\int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^{\frac{3}{2}}} dx;$$

(b)
$$\int_{-2}^2 \frac{1}{(1+x^2)^{\frac{3}{2}}} dx.$$

- (iii) (a) Show that

$$\int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^2} dx = \int_{\frac{1}{2}}^2 \frac{x^2}{(1+x^2)^2} dx = \frac{1}{2} \int_{\frac{1}{2}}^2 \frac{1}{1+x^2} dx,$$

and hence evaluate

$$\int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^2} dx.$$

- (b) Evaluate

$$\int_{\frac{1}{2}}^2 \frac{1-x}{x(1+x^2)^{\frac{1}{2}}} dx.$$

(i) Let $I = \int_a^b \frac{1}{(1+x^2)^{\frac{3}{2}}} dx$

With $x = \frac{1}{t}$, $dx = -\frac{1}{t^2} dt$, and $I = \int_{a^{-1}}^{b^{-1}} \frac{\frac{1}{t^2}}{(1+\frac{1}{t^2})^{\frac{3}{2}}} dt$

(As $a > 0$ & $b > 0$, t is defined throughout.)

$= \int_{a^{-1}}^{b^{-1}} \frac{-t}{(t^2+1)^{\frac{3}{2}}} dt$, which gives the required integral.

(ii)(a) From (i), integral $= \int_2^{\frac{1}{2}} \frac{-t}{(t^2+1)^{\frac{3}{2}}} dt$

$= \int_{\frac{1}{2}}^2 \frac{t}{(t^2+1)^{\frac{3}{2}}} dt$

Let $u = t^2$, so that $du = 2t dt$,

and the integral becomes $\int_{\frac{1}{4}}^4 \frac{\frac{1}{2}}{(u+1)^{\frac{3}{2}}} du$

$= \frac{1}{2} \left[\frac{1}{(-\frac{1}{2})} (u+1)^{-\frac{1}{2}} \right]_{\frac{1}{4}}^4$

$= - \left(\frac{1}{\sqrt{5}} - \frac{1}{\sqrt{\frac{5}{4}}} \right) = \frac{1}{\sqrt{5}}$

(b) As $\frac{1}{(1+x^2)^{\frac{3}{2}}}$ is an even function, integral $= 2 \int_0^2 \frac{1}{(1+x^2)^{\frac{3}{2}}} dx$

$= 2 \lim_{c \rightarrow 0} \int_c^2 \frac{1}{(1+x^2)^{\frac{3}{2}}} dx$ (*)

With $x = \frac{1}{t}$, $dx = -\frac{1}{t^2} dt$,

$$\begin{aligned}
 (*) &= 2 \lim_{d \rightarrow \infty} \int_d^{\frac{1}{2}} \frac{-\frac{1}{t^2}}{(1+\frac{1}{t^2})^{\frac{3}{2}}} dt \\
 &= 2 \lim_{d \rightarrow \infty} \int_{\frac{1}{2}}^{\infty} \frac{t}{(t^2+1)^{\frac{3}{2}}} dt \quad (**)
 \end{aligned}$$

Let $u = t^2$, so that $du = 2t dt$,

$$\begin{aligned}
 \text{and } (**) &\text{ becomes } 2 \lim_{d \rightarrow \infty} \int_{\frac{1}{4}}^{d^2} \frac{\frac{1}{2}}{(u+1)^{\frac{3}{2}}} du \\
 &= \lim_{d \rightarrow \infty} \left[\frac{1}{(-\frac{1}{2})} (u+1)^{-\frac{1}{2}} \right]_{\frac{1}{4}}^{d^2} \\
 &= -2 \left(0 - \frac{1}{\sqrt{\frac{5}{4}}} \right) = \frac{4}{\sqrt{5}} \text{ or } \frac{4\sqrt{5}}{5}
 \end{aligned}$$

$$\left[\int_{-2}^0 \frac{1}{(1+x^2)^{\frac{3}{2}}} dx = \lim_{c \rightarrow 0^-} \int_{-2}^c \frac{1}{(1+x^2)^{\frac{3}{2}}} dx \quad (\#)
 \right.$$

With $x = \frac{1}{t}$, $dx = -\frac{1}{t^2} dt$,

$$\begin{aligned}
 (\#) &= \lim_{d \rightarrow -\infty} \int_{-\frac{1}{2}}^d \frac{-\frac{1}{t^2}}{(1+\frac{1}{t^2})^{\frac{3}{2}}} dt \\
 &= \lim_{d \rightarrow -\infty} \int_d^{-\frac{1}{2}} \frac{t}{t^3(1+\frac{1}{t^2})^{\frac{3}{2}}} dt \quad (\#\#)
 \end{aligned}$$

For the domain of the integral, $t < 0$ and $\frac{t}{t^3(1+\frac{1}{t^2})^{\frac{3}{2}}}$ cannot be

written as $\frac{t}{(t^2+1)^{\frac{3}{2}}}$, as $t^3(1+\frac{1}{t^2})^{\frac{3}{2}} < 0$ whereas

$(t^2+1)^{\frac{3}{2}} > 0$ ($t^3 = t^{2 \times \frac{3}{2}}$, but this doesn't equal $(t^2)^{\frac{3}{2}}$ if $t < 0$;

in general, $t^{ab} = (t^a)^b$ is not valid for $t < 0$ unless both a & b are

integers)

Writing $T = -t$, so that $dT = -dt$,

$$(\#\#) \text{ becomes } \lim_{d \rightarrow -\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{-T}{-T^3(1+\frac{1}{T^2})^{\frac{3}{2}}} (-1) dT$$

$$= \lim_{D \rightarrow \infty} \int_{\frac{1}{2}}^D \frac{T}{T^3(1+\frac{1}{T^2})^{\frac{3}{2}}} dT$$

$$= \lim_{D \rightarrow \infty} \int_{\frac{1}{2}}^D \frac{T}{(T^2+1)^{\frac{3}{2}}} dT \quad (\#\#\#)$$

Let $u = T^2$, so that $du = 2T dT$,

$$\text{and } (\#\#\#) \text{ becomes } \lim_{D \rightarrow \infty} \int_{\frac{1}{4}}^{\infty} \frac{\frac{1}{2}}{(u+1)^{\frac{3}{2}}} du$$

$$= \frac{1}{2} \lim_{D \rightarrow \infty} \left[\frac{1}{(-\frac{1}{2})} (u+1)^{-\frac{1}{2}} \right]_{\frac{1}{4}}^{\infty}$$

$$= -\lim_{D \rightarrow \infty} \left(\frac{1}{\sqrt{D^2+1}} - \frac{1}{\sqrt{\frac{5}{4}}} \right)$$

$$= \frac{1}{\sqrt{\frac{5}{4}}} - 0 = \frac{2}{\sqrt{5}}]$$

(iii)(a) 1st Part

$$\text{Consider } J = \int_{\frac{1}{2}}^2 \frac{x^2}{(1+x^2)^2} dx$$

$$\text{Let } t = \frac{1}{x}, \text{ so that } dt = -\frac{1}{x^2} dx$$

$$\text{Then } J = \int_2^{\frac{1}{2}} \frac{\left(\frac{1}{t^2}\right)}{\left(1+\left(\frac{1}{t^2}\right)\right)^2} \left(-\frac{1}{t^2}\right) dt$$

$$= \int_{\frac{1}{2}}^2 \frac{1}{(t^2+1)^2} dt = \int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^2} dx$$

Thus, $\int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^2} dx = \int_{\frac{1}{2}}^2 \frac{x^2}{(1+x^2)^2} dx$, as required.

2nd Part

$$\text{Now, } \int_{\frac{1}{2}}^2 \frac{x^2}{(1+x^2)^2} dx = \int_{\frac{1}{2}}^2 \frac{x^2+1}{(1+x^2)^2} dx - \int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^2} dx$$

$$= \int_{\frac{1}{2}}^2 \frac{1}{1+x^2} dx - \int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^2} dx$$

$$\text{Hence, } \int_{\frac{1}{2}}^2 \frac{1}{1+x^2} dx = \int_{\frac{1}{2}}^2 \frac{x^2}{(1+x^2)^2} dx + \int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^2} dx$$

$$= 2 \int_{\frac{1}{2}}^2 \frac{x^2}{(1+x^2)^2} dx, \text{ from the 1st Part ;}$$

$$\text{so that } \int_{\frac{1}{2}}^2 \frac{x^2}{(1+x^2)^2} dx = \frac{1}{2} \int_{\frac{1}{2}}^2 \frac{1}{1+x^2} dx, \text{ as required.}$$

3rd Part

$$\text{Hence } \int_{\frac{1}{2}}^2 \frac{1}{(1+x^2)^2} dx = \frac{1}{2} \int_{\frac{1}{2}}^2 \frac{1}{1+x^2} dx$$

$$= \frac{1}{2} [\arctan x]_{\frac{1}{2}}^2$$

$$= \frac{1}{2} (\arctan 2 - \left(\frac{\pi}{2} - \arctan 2\right))$$

$$= \arctan 2 - \frac{\pi}{4}$$

(b) Write $I = \int_{\frac{1}{2}}^2 \frac{1-x}{x(1+x^2)^{\frac{1}{2}}} dx$

Let $t = \frac{1}{x}$, so that $dt = -\frac{1}{x^2} dx$

$$\text{Then } I = \int_2^{\frac{1}{2}} \frac{1 - \frac{1}{t}}{\frac{1}{t} \left(1 + \left(\frac{1}{t}\right)^2\right)^{\frac{1}{2}}} \left(-\frac{1}{t^2}\right) dt$$

$$= \int_{\frac{1}{2}}^2 \frac{t-1}{t(t^2+1)^{\frac{1}{2}}} dt = -I, \text{ so that } I = 0$$