STEP 2022, P2, Q10 - Solution (6 pages; 4/5/25)

10 (i) Show that, if a particle is projected at an angle α above the horizontal with speed u, it will reach height h at a horizontal distance s from the point of projection where

$$h = s\tan\alpha - \frac{gs^2}{2u^2\cos^2\alpha}.$$

The remainder of this question uses axes with the x- and y-axes horizontal and the z-axis vertically upwards. The ground is a sloping plane with equation $z = y\tan\theta$ and a road runs along the x-axis. A cannon, which may have any angle of inclination and be pointed in any direction, fires projectiles from ground level with speed u. Initially, the cannon is placed at the origin.

(ii) Let a point P on the plane have coordinates $(x\,,\ y\,,\ y\tan\theta)$. Show that the condition for it to be possible for a projectile from the cannon to land at point P is

$$x^2 + \left(y + \frac{u^2\tan\theta}{g}\right)^2 \leqslant \frac{u^4\sec^2\theta}{g^2}.$$

(iii) Show that the furthest point directly up the plane that can be reached by a projectile from the cannon is a distance

$$\frac{u^2}{g(1+\sin\theta)}$$

from the cannon.

How far from the cannon is the furthest point directly down the plane that can be reached by a projectile from it?

(iv) Find the length of road which can be reached by projectiles from the cannon.

The cannon is now moved to a point on the plane vertically above the y-axis, and a distance r from the road. Find the value of r which maximises the length of road which can be reached by projectiles from the cannon. What is this maximum length?

(i) Applying the suvat equation $'s = ut + \frac{1}{2}at^2{}'$ separately to horizontal and vertical motion:

At time t from projection, $s = ucos\alpha.t$ and $h = usin\alpha.t - \frac{1}{2}gt^2$

Eliminating t, $h = s.tan\alpha - \frac{1}{2}g(\frac{s}{ucos\alpha})^2$

$= s.tan\alpha - \frac{gs^2}{2u^2cos^2\alpha}$, as required.

(ii) The plane representing the ground can be thought of as the x-y plane, tilted by an angle θ about the x-axis (in the direction of the positive z-axis).

If the cannon is fired in the direction of P, then $s^2 = x^2 + y^2$, and we require the height of P above the x-y plane (ie $ytan\theta$) to be no greater than the maximum h possible. (We note that, for any such point P on the inclined plane, the projectile will be able to reach P, whilst remaining above the plane throughout its motion; ie the only consideration is whether the height of P exceeds the maximum possible.)

Now, $h = stan\alpha - \frac{gs^2}{2u^2}(tan^2\alpha + 1)$

$= -\frac{gs^2}{2u^2}(tan^2\alpha - \frac{2u^2}{gs^2}stan\alpha + 1)$

This is maximised when $tan^2\alpha - \frac{2u^2}{gs^2}stan\alpha + 1$ is minimised;

ie when $(tan\alpha - \frac{u^2}{gs})^2 - \frac{u^4}{g^2s^2} + 1$ is minimised.

This occurs when $tan\alpha - \frac{u^2}{gs} = 0$, and $h = -\frac{gs^2}{2u^2}(-\frac{u^4}{g^2s^2} + 1)$

So we require $ytan\theta \leq -\frac{gs^2}{2u^2}(-\frac{u^4}{g^2s^2} + 1)$ (*)

Now, the condition to be proved is

$x^2 + (y + \frac{u^2tan\theta}{g})^2 \leq \frac{u^4sec^2\theta}{g^2}$,

[The fact that the condition we are trying to demonstrate (C, say) is in a different form to (*) can suggest that we haven't derived the condition in the way that was intended by the question setter. It may be worth stopping to see if we have missed a more direct approach. But it might be the case that C is to be used for the next part of the question, and it was intended for (*) to be rearranged to produce C.]

and this is equivalent to

$x^2 + y^2 + \frac{u^4tan^2\theta}{g^2} + 2y.\frac{u^2tan\theta}{g} \leq \frac{u^4}{g^2}(tan^2\theta + 1)$;

or $s^2 + \frac{2yu^2tan\theta}{g} \leq \frac{u^4}{g^2}$, (**)

or $ytan\theta \leq \left(\frac{u^4}{g^2} - s^2\right).\frac{g}{2u^2} = -\frac{gs^2}{2u^2}(-\frac{u^4}{g^2s^2} + 1)$, which is (*),

as required.

(iii) **1st Part**

[It is easy to overlook the word 'directly' here.]

As the projectile is being fired **directly** up the plane (ie where the gradient of the plane is steepest), $x = 0$.

If the furthest point is $(0, y, ytan\theta)$, then the distance to that point from the cannon is $\sqrt{y^2 + (ytan\theta)^2} = ysec\theta$

As $x = 0$, the condition in (ii) becomes $(y + \frac{u^2 tan\theta}{g})^2 \leq \frac{u^4 sec^2\theta}{g^2}$

or $y + \frac{u^2 tan\theta}{g} \leq \frac{u^2 sec\theta}{g}$ (as both sides of this inequality are positive), and so the maximum value of $ysec\theta$ is

$\frac{u^2}{g}(sec\theta - tan\theta)sec\theta = \frac{u^2(1-sin\theta)}{gcos^2\theta}$

$= \frac{u^2(1-sin\theta)}{g(1-sin^2\theta)} = \frac{u^2}{g(1+sin\theta)}$, as required.

2nd Part

Let the furthest point directly down the plane be $(0, y, ytan\theta)$, where $y = -y'$, with $y' > 0$

The distance from the cannon is $\sqrt{y^2 + (ytan\theta)^2} = y'sec\theta$

and once again $(y + \frac{u^2 tan\theta}{g})^2 \leq \frac{u^4 sec^2\theta}{g^2}$

We want to find the smallest y (ie largest y') that satisfies this inequality.

This occurs when $y + \frac{u^2 tan\theta}{g} = -\frac{u^2 sec\theta}{g}$,

so that the required distance $y' = -y = \frac{u^2 tan\theta}{g} + \frac{u^2 sec\theta}{g}$

$= \frac{u^2}{g}(tan\theta + sec\theta)sec\theta = \frac{u^2(sin\theta+1)}{gcos^2\theta}$

$= \frac{u^2(1+sin\theta)}{g(1-sin^2\theta)} = \frac{u^2}{g(1-sin\theta)}$

[Check: This is larger than $\frac{u^2}{g(1+sin\theta)}$, which is to be expected, as gravity is assisting the motion.]

(iv) **1st Part**

With the projectile being fired in the direction of the road, $y = 0$, and the distance along the road is x.

The condition in (ii) becomes $x^2 + (\frac{u^2tan\theta}{g})^2 \leq \frac{u^4sec^2\theta}{g^2}$,

and so $x^2 \leq \frac{u^4}{g^2}(sec^2\theta - tan^2\theta) = \frac{u^4}{g^2}$,

and hence the maximum range along the road is $\frac{u^2}{g}$,

making a total length of $\frac{2u^2}{g}$, as the cannon can fire in either direction. [It is easy to overlook this!]

2nd Part

In (ii), we obtained the condition $ytan\theta \leq -\frac{gs^2}{2u^2}(-\frac{u^4}{g^2s^2} + 1)$ (*)

when the cannon was at the Origin, with P being at $(x, y, ytan\theta)$, so that $s^2 = x^2 + y^2$

With the cannon placed instead at the point $(0, rcos\theta, rsin\theta)$, this condition becomes $ytan\theta - rsin\theta \leq -\frac{gs^2}{2u^2}(-\frac{u^4}{g^2s^2} + 1)$, with $s^2 = x^2 + (y - rcos\theta)^2$ (and P still at $(x, y, ytan\theta)$),

so that $ytan\theta - rsin\theta \leq \frac{u^2}{2g} - \frac{g}{2u^2}[x^2 + (y - rcos\theta)^2]$

In order to maximise the distance along the road, we need to maximise x, with $y = 0$.

So $- rsin\theta \leq \frac{u^2}{2g} - \frac{g}{2u^2}[x^2 + (-rcos\theta)^2]$;

$\frac{g}{2u^2}[x^2 + r^2cos^2\theta\ ] \leq \frac{u^2}{2g} + rsin\theta$;

$x^2 + r^2cos^2\theta \leq \left(\frac{u^2}{2g} + rsin\theta\right)\frac{2u^2}{g}$;

$x^2 \leq \left(\frac{u^2}{2g} + rsin\theta\right)\frac{2u^2}{g} - r^2cos^2\theta$

$= \frac{u^4}{g^2} - (rcos\theta - tan\theta\,\frac{u^2}{g})^2 + tan^2\theta\,\frac{u^4}{g^2}$

Thus x is maximised when $rcos\theta - tan\theta\,\frac{u^2}{g} = 0$,

so that $r = \frac{tan\theta u^2}{gcos\theta}$ or $\frac{tan\theta sec\theta u^2}{g}$,

when $x^2 = \frac{u^4}{g^2}(1 + tan^2\theta) = \frac{u^4}{g^2}sec^2\theta$,

and $x = \frac{u^2 sec\theta}{g}$,

making a total length of $\frac{2u^2sec\theta}{g}$, as the cannon can fire in either direction.