STEP 2021, P3, Q3 - Solution (5 pages; 26/3/25)

3 (i) Let $I_n = \int_0^{\beta} (\sec x + \tan x)^n \, dx$, where n is a non-negative integer and $0 < \beta < \frac{\pi}{2}$.

For $n \geqslant 1$, show that

$$\frac{1}{2}(I_{n+1} + I_{n-1}) = \frac{1}{n}\big((\sec\beta + \tan\beta)^n - 1\big).$$

Show also that

$$I_n < \frac{1}{n}\big((\sec\beta + \tan\beta)^n - 1\big).$$

(ii) Let $J_n = \int_0^{\beta} (\sec x \cos\beta + \tan x)^n \, dx$, where n is a non-negative integer and $0 < \beta < \frac{\pi}{2}$.

For $n \geqslant 1$, show that

$$J_n < \frac{1}{n}\big((1 + \tan\beta)^n - \cos^n\beta\big).$$

(i) **1st Part**

$\frac{1}{2}(I_{n+1} + I_{n-1})$

$= \frac{1}{2}[\int_0^\beta (secx + tanx)^{n+1}dx + \int_0^\beta (secx + tanx)^{n-1}dx]$

$= \frac{1}{2}\int_0^\beta (secx + tanx)^{n-1}(sec^2x + tan^2x + 2secxtanx + 1)dx$

$= \int_0^\beta (secx + tanx)^{n-1}(sec^2x + secxtanx)dx$

Now, $\frac{d}{dx}\left[\frac{1}{n}(secx + tanx)^n\right]$

$= (secx + tanx)^{n-1}[-(cosx)^{-2}(-sinx) + sec^2x]$

$= (secx + tanx)^{n-1}[sec^2x + secxtanx]$

And $\frac{1}{n}(\sec(0) + \tan(0))^n = \frac{1}{n}$,

so that $\int_0^\beta (secx + tanx)^{n-1}(sec^2x + secxtanx)dx$

$= \left[\frac{1}{n}(secx + tanx)^n\right]_0^\beta$

$= \frac{1}{n}(sec\beta + tan\beta)^n - \frac{1}{n}$, as required.

$= \frac{1}{n}[(sec\beta + tan\beta)^n - 1]$

2nd Part

Method 1

Suppose instead that $I_n \geq \frac{1}{n}[(sec\beta + tan\beta)^n - 1]$

Then $\frac{1}{2}(I_{n+1} + I_{n-1}) \geq \frac{1}{2n}[(sec\beta + tan\beta)^{n+1} - 1]$

$+\frac{1}{2n}[(sec\beta + tan\beta)^{n-1} - 1]$

$= \frac{1}{2n}(sec\beta + tan\beta)^{n-1}[(sec\beta + tan\beta)^2 + 1]$

$= \frac{1}{2n}(sec\beta + tan\beta)^{n-1}[sec^2\beta + tan^2\beta + 2sec\beta tan\beta + 1]$

$= \frac{1}{n}(sec\beta + tan\beta)^{n-1}(sec^2\beta + sec\beta tan\beta)$

$= \frac{1}{n}(sec\beta + tan\beta)^n sec\beta > \frac{1}{n}(sec\beta + tan\beta)^n$, as $0 < \beta < \frac{\pi}{2}$ (*)

But, from the 1st Part, $\frac{1}{2}(I_{n+1} + I_{n-1}) = \frac{1}{n}[(sec\beta + tan\beta)^n - 1]$

$< \frac{1}{n}[(sec\beta + tan\beta)^n$, which contradicts (*).

Hence $I_n < \frac{1}{n}[(sec\beta + tan\beta)^n - 1]$

Method 2

From the 1st Part, the result to prove is equivalent to

$I_n < \frac{1}{2}(I_{n+1} + I_{n-1})$ or $I_{n+1} + I_{n-1} - 2I_n > 0$

And $I_{n+1} + I_{n-1} - 2I_n =$

$\int_0^\beta (secx + tanx)^{n+1} + (secx + tanx)^{n-1} - 2(secx + tanx)^n\, dx$

$= \int_0^\beta (secx + tanx)^{n-1}(sec^2x + tan^2x + 2secxtanx + 1$

$-2secx - 2tanx)\, dx$

$= 2\int_0^\beta (secx + tanx)^{n-1}(sec^2x + secxtanx - secx - tanx)dx$

$= 2\int_0^\beta (secx + tanx)^{n-1}(\sec x + tanx)(secx - 1)\, dx$

$= 2\int_0^{\beta}(secx + tanx)^n(secx - 1)dx > 0$, as required,

as both $secx + tanx$ and $secx - 1$ are positive for $0 < x < \beta < \frac{\pi}{2}$

(ii) [It isn't clear what approach the question setter has in mind here. Possible options are:

(a) Applying exactly the same method; ie starting by showing that

$\frac{1}{2}(J_{n+1} + J_{n-1}) = \frac{1}{n}((1 + tanx)^n - cos^n x)$ [This gives the

following for the LHS:

$\frac{1}{2}[\int_0^{\beta}(secxcos\beta + tanx)^{n+1}dx + \int_0^{\beta}(secxcos\beta + tanx)^{n-1}dx]$

$= \frac{1}{2}\int_0^{\beta}(secxcos\beta + tanx)^{n-1}(sec^2xcos^2\beta + tan^2x$

$+2secxsinx + 1)dx$, which isn't very promising.]

(b) Modifying the method in some way that takes account of the differences between J_n and I_n. Nothing obvious springs to mind.

(c) Using the result of Part (i) in some way; eg by making a substitution. Again, nothing obvious springs to mind.

(d) Using an idea that was involved in answering Part (i).

One idea was that of showing that an integral had a positive value. In order to use this we will need to be able to write

$\frac{1}{n}((1 + tanx)^n - cos^n x)$ as in integral (K_n, say), and show that

$K_n - J_n > 0$

Another idea was using the fact that certain components of the integrand were positive.

(e) Using an idea prompted by a difference between J_n and I_n.

One such idea that emerges later on is that $cos\beta < cosx$ for

$0 < x < \beta$, and this enables the awkward $secxcos\beta$ to be converted into $secxcosx = 1$]

Consider $\frac{d}{dx}[\frac{1}{n}((1 + tanx)^n - cos^n x)]$

$= (1 + tanx)^{n-1}sec^2x - cos^{n-1}x(-sinx)$

Let $K_n = \int_0^\beta (1 + tanx)^{n-1}sec^2x + cos^{n-1}xsinxdx$

$= \left[\frac{1}{n}((1 + tanx)^n - cos^n x)\right]_0^\beta$

$= \frac{1}{n}((1 + tan\beta)^n - cos^n\beta) - 0$

Then the result to be proved is that $K_n - J_n > 0$

Now, $K_n - J_n = \int_0^\beta (1 + tanx)^{n-1}sec^2x + cos^{n-1}xsinx$

$-(secxcos\beta + tanx)^{n+1}dx$

$> \int_0^\beta (1 + tanx)^{n-1}sec^2x + cos^{n-1}xsinx$

$-(secxcosx + tanx)^{n+1}dx$,

as $x < \beta \Rightarrow cosx > cos\beta$ (for $0 < \beta < \frac{\pi}{2}$),

$= \int_0^\beta (1 + tanx)^{n-1}sec^2x + cos^{n-1}xsinx - (1 + tanx)^{n+1}dx$,

$= \int_0^\beta (1 + tanx)^{n-1}(tan^2x + 1) + cos^{n-1}xsinx$

$-(1 + tanx)^{n+1}dx$

$= \int_0^\beta (1 + tanx)^{n-1} + cos^{n-1}xsinx > 0$, as required,

as $1 + tanx$, $cosx$ & $sinx$ are all positive for $0 < x < \beta < \frac{\pi}{2}$