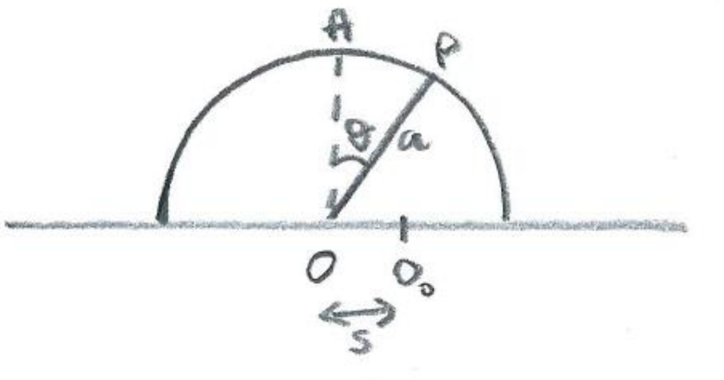
STEP 2019, P3, Q9 - Solution (3 pages; 22/7/20)

(i) **1st part**



Referring to the diagram, $\underline{r} = (a sin\theta - s)\underline{i} + a cos\theta \underline{j}$

2nd part

$\underline{\dot{r}} = (a cos\theta . \dot{\theta} - \dot{s})\underline{i} - a sin\theta . \dot{\theta}\underline{j}$

or $(a\dot{\theta}cos\theta - \dot{s})\underline{i} - a\dot{\theta}sin\theta\underline{j}$, as required.

3rd part

By conservation of momentum in the $\underline{i}$ direction (assuming that the impulse causing the displacement is negligible)

[There is an external force (gravity) acting in the (negative) $\underline{j}$ direction, so that there is no conservation of momentum.]

$M(-\dot{s}) + m(a\dot{\theta}cos\theta - \dot{s}) = 0$

$\Rightarrow ma\dot{\theta}cos\theta = \dot{s}(M + m)$

$\Rightarrow \dot{s} = \frac{m}{m+M}\, a\dot{\theta}cos\theta = (1 - k)a\dot{\theta}cos\theta$,

where $k = \frac{M}{m+M}$, as required.

4th part

So $\underline{\dot{r}} = \left(a\dot{\theta}cos\theta - \dot{s}\right)\underline{i} - a\dot{\theta}sin\theta\underline{j}$

$= \left(a\dot{\theta}cos\theta - (1-k)a\dot{\theta}cos\theta\right)\underline{i} - a\dot{\theta}sin\theta\underline{j}$

$= a\dot{\theta}(kcos\theta\underline{i} - sin\theta\underline{j})$, as required.

(ii) By conservation of energy,

increase in KE = reduction in PE (of particle),

so that $\frac{1}{2}M\dot{s}^2 + \frac{1}{2}m|\underline{\dot{r}}|^2 = mga(1-cos\theta)$

$\Rightarrow M\left[(1-k)a\dot{\theta}cos\theta\right]^2 + m\left(a\dot{\theta}\right)^2[(kcos\theta)^2 + (-sin\theta)^2]$

$= 2mga(1-cos\theta)$

$\Rightarrow a\dot{\theta}^2\left\{\frac{M}{m}(1-k)^2cos^2\theta + k^2cos^2\theta + sin^2\theta\right\}$

$= 2g(1-cos\theta)$ (A)

Now, $k = \frac{M}{m+M} = \frac{1}{\left(\frac{m}{M}\right)+1}$, so that $\frac{m}{M} + 1 = \frac{1}{k}$

and $\frac{m}{M} = \frac{1-k}{k}$, so that $\frac{M}{m} = \frac{k}{1-k}$

and hence $\frac{M}{m}(1-k)^2 + k^2 = k(1-k) + k^2 = k$

Then (A) $\Rightarrow$ $a\dot{\theta}^2(kcos^2\theta + sin^2\theta) = 2g(1-cos\theta)$, as required.

(iii) **1st part**

[The suggested approach of considering the component of $\underline{\ddot{r}}$ parallel to the vector $sin\theta\underline{i} + kcos\theta\underline{j}$ is arguably more of a hindrance than a help. $\underline{\ddot{r}} = -g\underline{j}$ is the key idea, and it isn't necessary to consider the component). Also, there is no explanation in the official sol'n as to why considering this component might be a good idea (it just happens to give the required result). The vector $sin\theta\underline{i} + kcos\theta\underline{j}$ is perpendicular to $\underline{\dot{r}}$, but it isn't obvious why this is relevant (if it is).]

$\underline{\dot{r}} = a\dot{\theta}(kcos\theta\underline{i} - sin\theta\underline{j})$

$\Rightarrow \underline{\ddot{r}} = a\ddot{\theta}\left(kcos\theta\underline{i} - sin\theta\underline{j}\right) + a\dot{\theta}(-ksin\theta.\dot{\theta}\underline{i} - cos\theta.\dot{\theta}\underline{j})$

When the particle loses contact, $\underline{\ddot{r}} = -g\underline{j}$

$\Rightarrow a\ddot{\theta}kcos\alpha - a\dot{\theta}^2ksin\alpha = 0$

$\Rightarrow \ddot{\theta}cos\alpha - \dot{\theta}^2sin\alpha = 0$

and $a\ddot{\theta}sin\alpha + a\dot{\theta}^2cos\alpha = g$

Eliminating $\ddot{\theta}$, $a(\frac{\dot{\theta}^2sin\alpha}{cos\alpha})sin\alpha + a\dot{\theta}^2cos\alpha = g$

$\Rightarrow a\dot{\theta}^2(sin^2\alpha + cos^2\alpha) = gcos\alpha$

$\Rightarrow a\dot{\theta}^2 = gcos\alpha$, as required.

2nd part

Substituting for $a\dot{\theta}^2$ in the result shown in (ii),

$gcos\alpha(kcos^2\alpha + sin^2\alpha) = 2g(1 - cos\alpha)$,

$\Rightarrow kcos^3\alpha + cos\alpha(1 - cos^2\alpha) - 2 + 2cos\alpha = 0$

$\Rightarrow (k-1)cos^3\alpha + 3cos\alpha - 2 = 0$

3rd part

$\Rightarrow 3cos\alpha - 2 = (1-k)cos^3\alpha > 0$ (as $\alpha < \frac{\pi}{2}$)

$\Rightarrow cos\alpha > \frac{2}{3}$, as required.