STEP 2019, P3, Q3 - Solution (5 pages; 22/7/20)

[In the context of transformations, it is the columns of $\underline{A}$ that are significant, rather than the rows. So arguably the labelling $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is preferable to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.]

(i) **1st part**

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (A)

$\Rightarrow (a-1)x + by = 0$ & $cx + (d-1)y = 0$

$\Rightarrow (a-1)(d-1)xy + b(d-1)y^2 = 0$ & $bcxy + b(d-1)y^2 = 0$

and subtracting the 2nd eq'n from the 1st gives

$(a-1)(d-1)xy - bcxy = 0$,

so that $((a-1)(d-1) - bc)xy = 0$, as required.

2nd part

$x = 0, y = 0$ will always be a solution to (A).

So, in order for there to be a line of invariant points, there must be more than one solution to (A), and hence $\begin{vmatrix} a-1 & b \\ c & d-1 \end{vmatrix} = 0$;

ie $(a-1)(d-1) - bc = 0$,

so that $(a-1)(d-1) = bc$, as required.

[The result for the 1st part can of course be obtained from this.]

3rd part

$\underline{A}$ has to be the identity matrix. This is a sufficient condition for the existence of a line of invariant points that doesn't pass through the Origin, as every line is an invariant line. To show that it is necessary, suppose that the line of invariant points is

$y = mx + k$, where $k \neq 0$ (the other possibility, $x = k$ is considered next).

Then, from (A), $\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix}\begin{pmatrix} x \\ mx+k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, for some m & k, and all values of x.

Thus $(a-1)x + b(mx+k) = 0$

and $cx + (d-1)(mx+k) = 0$

Then, equating coefficients of x & x^0,

$a - 1 + bm = 0; bk = 0; c + (d-1)m = 0; (d-1)k = 0$

From the 2nd eq'n, $b = 0$, as $k \neq 0$

Similarly, from the 4th eq'n, $d = 1$

Then, from the 1st eq'n, $a = 1$, and from the 3rd eq'n: $c = 0$

Thus $\underline{A}$ is the identity matrix.

If instead $x = k$ (where $k \neq 0$),

$\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix}\begin{pmatrix} k \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, for all values of y

so that $(a-1)k + by = 0$ & $ck + (d-1)y = 0$

Then, equating coefficients of y & y^0,

$(a-1)k = 0; b = 0; ck = 0; d - 1 = 0$

and as before $a = 1, b = 0, c = 0$ & $d = 1$

(ii) Suppose that $(a-1)(d-1) = bc$, with $b \neq 0$.

Then $\underline{A}$ is not the identity matrix, so that any lines of invariant points must pass through the Origin.

Suppose that $y = mx$ is a line of invariant points (the other possibility, $x = 0$ can be considered if necessary).

Then, from (A), $\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for some m and all x

So $(a-1)x + bmx = 0 \;\; \& \;\; cx + (d-1)mx = 0 \quad$ (B)

Then, equating coefficients of x,

$a - 1 + bm = 0 \; \& \; c + (d-1)m = 0$,

so that $m = \frac{1-a}{b}$ and also $m = \frac{c}{1-d}$

Then a line of invariant points exists when $d \neq 1$ (as $b \neq 0$), and provided $\frac{1-a}{b} = \frac{c}{1-d}$; ie $(1-a)(1-d) = bc$, or

$(a-1)(d-1) = bc$, and this is satisfied.

If $d = 1$ then $(a-1)(d-1) = bc \Rightarrow c = 0$, as $b \neq 0$

Then (B) $\Rightarrow m = \frac{1-a}{b}$

Suppose instead that $(a-1)(d-1) = bc$, with $b = 0$.

Then either $a = 1 \; or \; d = 1$. (C)

If $\underline{A}$ is the identity matrix, then a line of invariant points exists.

If $\underline{A}$ is not the identity matrix, and $y = mx$ is a line of invariant points, then $\begin{pmatrix} a-1 & 0 \\ c & d-1 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for some m and all x

Then $(a-1)x = 0 \;\; \& \;\; cx + (d-1)mx = 0 \;$ (D)

Equating coefficients of x,

$a = 1; c + (d-1)m = 0$

Thus a line of invariant points exists if $a = 1$ & $d \neq 1$

(when $m = \frac{c}{1-d}$).

If $a \neq 1$, so that $d = 1$ (from (C), then we can consider $x = 0$ as a possible line of invariant points.

Then $\begin{pmatrix} a-1 & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all y, and this is satisfied.

Summary

$\boldsymbol{b \neq 0}$ **:** line of invariant points is $y = \frac{1-a}{b}x$ (or $\frac{c}{1-d}x$)

$\boldsymbol{b = 0}$

If $\underline{A}$ is the identity matrix, then all lines are lines of invariant points.

If $\underline{A}$ is not the identity matrix:

$a = 1$ & $d \neq 1$: line of invariant points is $y = \frac{1-a}{b}x$ (or $\frac{c}{1-d}x$)

$a \neq 1$ & $d = 1$: line of invariant points is $x = 0$

(iii) A point on the invariant line transforms to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ mx+k \end{pmatrix} = \begin{pmatrix} ax+bmx+bk \\ cx+dmx+dk \end{pmatrix}$$

In order for this point to lie on the invariant line,

$cx + dmx + dk = m(ax + bmx + bk) + k$ (for all x)

Equating coefficients of x & x^0,

$c + dm = am + bm^2 \Rightarrow bm^2 + (a-d)m - c = 0$

& $dk = mbk + k \Rightarrow d = mb + 1$, as $k \neq 0$

$\Rightarrow m = \frac{d-1}{b}$

Then, substituting in the quadratic in m,

$b(\frac{d-1}{b})^2 + (a-d)(\frac{d-1}{b}) - c = 0$

$\Rightarrow (d-1)^2 + (a-d)(d-1) - bc = 0$

$\Rightarrow (d-1)(d-1+a-d) = bc$

$\Rightarrow (d-1)(a-1) = bc$, as required.