STEP 2019, P2, Q2 - Solution (3 pages; 9/7/20)

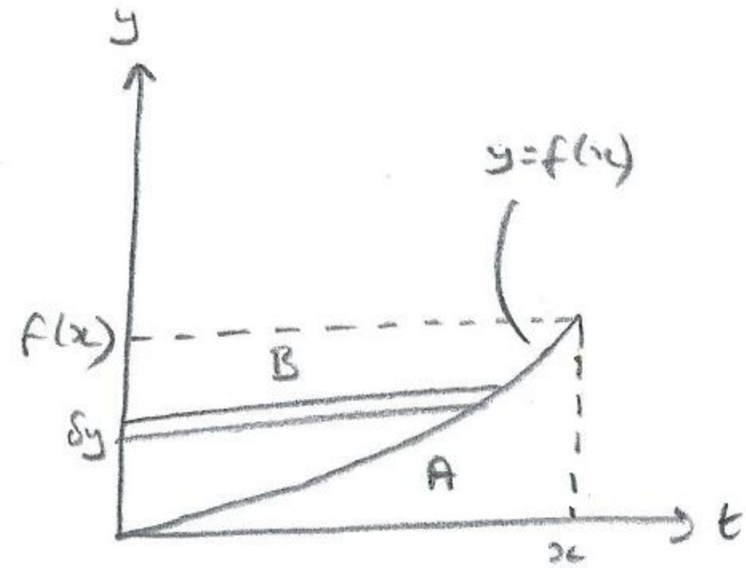
1st part

[To show algebraically: let $y = f(t)$, so that $dy = f'(t)dt$

and $\int_0^{f(x)} f^{-1}(y)dy = \int_0^x tf'(t)dt$

$= [tf(t)]_0^x - \int_0^x f(t)dt$, by Parts

Then $\int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(y)dy = xf(x)$, as required.]



$\int_0^x f(t)dt$ is area A in the diagram

And on the curve $y = f(t), f^{-1}(y) = t$,

so that area B is $\lim_{\delta y \to 0} \sum_{y=0}^{f(x)} t\delta y = \int_0^{f(x)} f^{-1}(y)dy$

And area A + area B $= xf(x)$, as required.

(i) 1st part

$(g(t))^3 + g(t) = t \Rightarrow (g(0))^3 + g(0) = 0$

$\Rightarrow g(0)\{(g(0))^2 + 1\} = 0$

Then, as $(g(0))^2 + 1 > 0$, $g(0) = 0$, as required.

2nd part

$(g(t))^3 + g(t) = t \Rightarrow 3(g(t))^2 g'(t) + g'(t) = 1$

$\Rightarrow g'(t)\left\{3(g(t))^2 + 1\right) = 1$

$\Rightarrow g'(t) = \frac{1}{\left\{3(g(t))^2+1\right)} > 0$, as $3(g(t))^2 + 1 > 0$

3rd part

Use the initial result, with $f(t) = g(t)$ and $x = 2$

Let $u(y) = g^{-1}(y)$, so that $g(u) = y$

Then $(g(t))^3 + g(t) = t \Rightarrow\ y^3 + y = u$

Also, $(g(2))^3 + g(2) = 2$, and $g(2) = 1$ is a solution, by inspection.

Consider the cubic $y = x^3 + x - 2$

$\frac{dy}{dx} = 3x^2 + 1 > 0$, and so there is only one solution to

$x^3 + x - 2 = 0$, and hence $g(2) = 1$ is the only solution.

Then $\int_0^x f(t)dt + \int_0^{f(x)} f^{-1}(y)dy = xf(x)$

$\Rightarrow \int_0^2 g(t)dt + \int_0^1 y^3 + y\, dy = 2g(2) = 2$

$\Rightarrow \int_0^2 g(t)dt = 2 - \left[\frac{1}{4}y^4 + \frac{1}{2}y^2\right]\begin{matrix}1\\0\end{matrix} = 2 - \left(\frac{1}{4} + \frac{1}{2}\right) = \frac{5}{4}$

(ii) Let $h(t) = k(t) + a$, so that $(k(t) + a)^3 + k(t) + a = t + 2$,

and we want $k(0) = 0$, so that $a^3 + a = 2$

So let $a = 1$

Then $3(k(t)+1)^2k'(t) + k'(t) = 1$,

so that $k'(t) = \frac{1}{3(k(t)+1)^2+1} > 0$

Let $u(y) = k^{-1}(y)$, so that $k(u) = y$

Then $(k(t)+1)^3 + k(t) + 1 = t + 2 \Rightarrow (y+1)^3 + y + 1 = u + 2$

$\Rightarrow u = y^3 + 3y^2 + 4y$

Also, $(k(8)+1)^3 + k(8) + 1 = 8 + 2$, giving $k(8) = 1$

Then, with $f(t) = k(t)$ and $x = 8$,

$\int_0^8 k(t)dt + \int_0^1 y^3 + 3y^2 + 4y\, dy = 8k(8) = 8$

so that $\int_0^8 h(t) - 1\, dt + \left[\frac{1}{4}y^4 + y^3 + 2y^2\right]_0^1 = 8$

$\Rightarrow \int_0^8 h(t)\, dt = 8 + [t]_0^8 - (\frac{1}{4} + 1 + 2)$

$= 8 + 8 - \frac{13}{4}$

$= \frac{64-13}{4} = \frac{51}{4}$

[Note: In the official sol'ns, $h(t) = g(t+2)$, and $h(0) \neq 0$; but the initial result can be modified, by changing the lower limit of the 2nd integral to $h(0)$. This can be seen from a new sketch - see below.]