STEP 2019, Paper 2, Q3 Solution (4 pages; 4/6/25)

3 For any two real numbers x_1 and x_2, show that

$$|x_1 + x_2| \leqslant |x_1| + |x_2|.$$

Show further that, for any real numbers $x_1, x_2, \ldots, x_n$,

$$|x_1 + x_2 + \cdots + x_n| \leqslant |x_1| + |x_2| + \cdots + |x_n|.$$

(i) The polynomial f is defined by

$$\mathrm{f}(x) = 1 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + x^n$$

where the coefficients are real and satisfy $|a_i| \leqslant A$ for $i = 1, 2, \ldots, n-1$, where $A \geqslant 1$.

(a) If $|x| < 1$, show that

$$|\mathrm{f}(x) - 1| \leqslant \frac{A|x|}{1-|x|}.$$

(b) Let ω be a real root of f, so that $\mathrm{f}(\omega) = 0$. In the case $|\omega| < 1$, show that

$$\frac{1}{1+A} \leqslant |\omega| \leqslant 1 + A. \qquad (*)$$

(c) Show further that the inequalities $(*)$ also hold if $|\omega| \geqslant 1$.

(ii) Find the integer root or roots of the quintic equation

$$135x^5 - 135x^4 - 100x^3 - 91x^2 - 126x + 135 = 0.$$

1st & 2nd Parts

Result to prove :

$-(|x_1| + |x_2| + \cdots + |x_n|) \leq x_1 + x_2 + \cdots + x_n$

$\leq |x_1| + |x_2| + \cdots + |x_n|$

The right-hand inequality is immediately clear, as either $|x_i| = x_i$, if $x_i \geq 0$; or $|x_i| > x_i$, if $x_i < 0$.

For the left-hand inequality: If $x_i \geq 0$, then $-|x_i| < x_i$; and if

$x_i < 0$, then $-|x_i| = x_i$. And the result follows from this.

(i)(a) $|f(x) - 1| = |a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + x^n|$

$\leq |a_1 x| + |a_2 x^2| + \cdots + |a_{n-1}x^{n-1}| + |x^n|$, from the 2nd Part;

$\leq |Ax| + |Ax^2| + \cdots + |Ax^{n-1}| + |x^n|$

$= A(|x| + |x|^2 + \cdots + |x|^{n-1}) + A|x|^n$, as $A > 1$

$= A|x|.\frac{1-|x^n|}{1-|x|}$

$\leq A|x|.\frac{1}{1-|x|}$ QED

(b) From (a), $|f(\omega) - 1| \leq A|\omega|.\frac{1}{1-|\omega|}$,

so that $|0 - 1| \leq A|\omega|.\frac{1}{1-|\omega|}$,

and hence $1 - |\omega| \leq A|\omega|$, as $1 - |\omega| > 0$;

so that $|\omega|(A + 1) \geq 1$, and $|\omega| \geq \frac{1}{1+A}$

Also, as $|\omega| < 1$, $|\omega| < 1 + A$, as $A \geq 1 > 0$, and so it follows that $\frac{1}{1+A} \leq |\omega| \leq 1 + A$

(c) If $|\omega| \geq 1$, then $\frac{1}{1+A} < 1 \leq |\omega|$

To prove that $|\omega| \leq 1 + A$:

Write

$f(x) = x^n(1 + a_{n-1}\left(\frac{1}{x}\right) + a_{n-2}\left(\frac{1}{x}\right)^2 + \cdots + a_1\left(\frac{1}{x}\right)^{n-1} + \left(\frac{1}{x}\right)^n)$

and write

$g(y) = 1 + a_{n-1}y + a_{n-2}y^2 + \cdots + a_1 y^{n-1} + y^n$

Then, as $f(\omega) = 0$ (and $\omega \neq 0$), it follows that $g\left(\frac{1}{\omega}\right) = 0$.

Then (i)(a) can be applied to $g(y)$ (in place of $f(x)$),

to give:

$|g(y) - 1| \leq A|y|.\frac{1}{1-|y|}$ (as the coefficients are as before, though in reverse order)

Setting $y = \frac{1}{\omega}$, and noting that $|y| < 1$ in the case where $|\omega| > 1$:

$|0 - 1| \leq A(\frac{1}{|\omega|}).\frac{1}{1-\frac{1}{|\omega|}}$,

so that, for $|\omega| > 1$, $1 \leq \frac{A}{|\omega|-1}$,

and $|\omega| - 1 \leq A$ (as $|\omega| - 1 > 0$)

ie $|\omega| \leq 1 + A$ for $|\omega| > 1$

Finally, for $|\omega| = 1$, it is again true that $|\omega| \leq 1 + A$ (as $A > 0$).

Thus $\frac{1}{1+A} \leq |\omega| \leq 1 + A$ for $|\omega| \geq 1$ also.

(ii) Dividing through by 135 produces a polynomial equation $f(x) = 0$ as in (i), where the coefficients satisfy $|a_i| \leq 1$.

Thus, from (*), any real roots ω will satisfy

$\frac{1}{1+1} \leq |\omega| \leq 1 + 1$, or $\frac{1}{2} \leq |\omega| \leq 2$;

and so the only possible integer roots are ± 1 or ± 2

As the constant term of the original polynomial equation is 135; an odd number, it will not be possible for an integer root to be even (as all the other terms, $a_i x^i$ will be even when x is even).

Substituting $x = 1$ and $x = -1$ into the original polynomial equation, we find that only $x = -1$ is a root, and so this is the only integer root. [There is a slip in the Official solution, which presumably meant to say that $f(-1) = 0$.]