STEP 2018, P3, Q8 - Solution (2 pages; 28/5/19)

(i) Let $y = x^{-1}$, so that $dy = -x^{-2}dx = -y^2dx$,

and $\frac{f(x^{-1})}{1+x}\ dx = \frac{f(y)(-\frac{dy}{y^2})}{1+\frac{1}{y}} = \frac{-f(y)dy}{y(y+1)}$

And when $x = 0, y = \infty$ (strictly speaking, I is an improper integral and we should consider an upper limit of $y = a$, and investigate $a \to \infty$, but we are allowed to ignore issues of convergence).

And when $x = 1, y = 1$.

So $I = \int_0^1 \frac{f(x^{-1})}{1+x}\ dx = \int_\infty^1 \frac{-f(y)dy}{y(y+1)} = \int_1^\infty \frac{f(y)dy}{y(y+1)}$

$= \int_1^2 \frac{f(y)dy}{y(y+1)} + \int_2^3 \frac{f(y)dy}{y(y+1)} + \cdots\ = \sum_{n=1}^\infty \int_n^{n+1} \frac{f(y)dy}{y(1+y)}$

$I = \sum_{n=1}^\infty \{\int_n^{n+1} \frac{f(y)}{y} dy - \int_n^{n+1} \frac{f(y)}{1+y}\ dy\}$

Let $y = 1 + u$ in the 1st integral.

Then $I = \sum_{n=1}^\infty \{\int_{n-1}^{n} \frac{f(u)}{1+u} du - \int_n^{n+1} \frac{f(y)}{1+y}\ dy\}$, as $f(1+u) = f(u)$

$= \int_0^\infty \frac{f(x)}{1+x}\ dx - \int_1^\infty \frac{f(x)}{1+x}\ dx = \int_0^1 \frac{f(x)}{1+x}\ dx$

(ii) Let $f(x) = \{x\}$, so that $f(x+1) = f(x)$

and from (i): $\int_0^1 \frac{\{x^{-1}\}}{1+x}\ dx = \int_0^1 \frac{\{x\}}{1+x}\ dx = \int_0^1 \frac{x}{1+x}\ dx$,

as $\{x\} = x$ for $x \in [0,1)$

And $\int_0^1 \frac{x}{1+x}\ dx = \int_0^1 \frac{x+1}{1+x} - \frac{1}{1+x}\ dx = [x - \ln(1+x)]_0^1$

$= (1 - ln2) - (0 - 0) = 1 - ln2$

Let $f(x) = \{2x\}$, so that $f(x+1) = \{2x+2\} = \{2x\} = f(x)$

Then from (i): $\int_0^1 \frac{\{2x^{-1}\}}{1+x}\, dx = \int_0^1 \frac{\{2x\}}{1+x}\, dx$

$= \int_0^{0.5} \frac{\{2x\}}{1+x}\, dx + \int_{0.5}^1 \frac{\{2x\}}{1+x}\, dx$

$= \int_0^{0.5} \frac{2x}{1+x}\, dx + \int_{0.5}^1 \frac{2x-1}{1+x}\, dx$

$= \int_0^1 \frac{2x}{1+x}\, dx - \int_{0.5}^1 \frac{1}{1+x}\, dx$

$= 2\int_0^1 \frac{x}{1+x}\, dx - [\ln(1+x)]_{0.5}^{1}$

$= 2(1 - ln2) - (ln2 - \ln\left(\frac{3}{2}\right))$, from the previous part

$= 2 - 3ln2 + (ln3 - ln2) = 2 + ln3 - 4ln2$