STEP 2018, P3, Q2 - Solution (3 pages; 23/5/19)

(i) $\frac{dy_n}{dx} = (-1)^n\left(-\frac{1}{z^2}\right)\frac{dz}{dx}\,\frac{d^nz}{dx^n} + (-1)^n(\frac{1}{z})\frac{d^{n+1}z}{dx^{n+1}}$ (1)

and $\left(-\frac{1}{z^2}\right)\frac{dz}{dx} = \frac{1}{z}\left(-\frac{1}{z}\right)(-2x)z = 2x(\frac{1}{z})$,

and $(-1)^n = -(-1)^{n+1}$,

so that $(1) = 2xy_n - y_{n+1}$

(ii) From (i), the required result will be true if $\frac{dy_n}{dx} = 2ny_{n-1}$

When $n = 1$, RHS $= 2y_0 = 2$

Now $y_1 = -\frac{1}{z}\,\frac{dz}{dx} = -\frac{1}{z}(-2x)z = 2x$, so that $LHS = \frac{dy_1}{dx} = 2$

Thus the result is true for $n = 1$.

Assume that the result is true for $n = k$, so that

$y_{k+1} = 2xy_k - 2ky_{k-1}$ (2)

The aim is to prove that it is then true that

$y_{k+2} = 2xy_{k+1} - 2(k+1)y_k$ (3) (the result for $n = k+1$)

From (i), $y_{k+2} = 2xy_{k+1} - \frac{dy_{k+1}}{dx}$ (4)

and by (2), $\frac{dy_{k+1}}{dx} = 2y_k + 2x\frac{dy_k}{dx} - 2k\frac{dy_{k-1}}{dx}$,

which by (i) equals $2y_k + 2x(2xy_k - y_{k+1}) - 2k(2xy_{k-1} - y_k)$

$= y_k(2 + 4x^2 + 2k) - 2xy_{k+1} - 4kxy_{k-1}$

which by (2) equals

$y_k(2 + 4x^2 + 2k) - 2x(2xy_k - 2ky_{k-1}) - 4kxy_{k-1}$

$= y_k(2 + 2k)$,

so that (4) becomes $y_{k+2} = 2xy_{k+1} - 2y_k(1+k)$,

which is the required result (3).

So if the result is true for $n = k$, then it is true for $n = k + 1$.

As the result is true for $n = 1$, it is therefore true for $n = 2,3, \dots$, and hence all integer $n \geq 1$, by the principle of induction.

$y_{n+1}^2 - y_n y_{n+2} - 2n(y_n^2 - y_{n-1}y_{n+1}) - 2y_n^2$

$= 4x^2y_n^2 + 4n^2y_{n-1}^2 - 8xny_ny_{n-1}$

$-y_n(2xy_{n+1} - 2(n+1)y_n) - 2(n+1)y_n^2 + 2ny_{n-1}y_{n+1}$

$= 4x^2y_n^2 + 4n^2y_{n-1}^2 - 8xny_ny_{n-1}$

$-2xy_n(2xy_n - 2ny_{n-1}) + 2ny_{n-1}(2xy_n - 2ny_{n-1})$

$= 0$, which proves the required result.

(iii) To carry out a proof by induction,

result to prove: $y_1^2 - y_0y_2 > 0$ (when $n = 1$)

By definition, $y_1 = -\frac{1}{z}\frac{dz}{dx} = -\frac{1}{z}(-2x)z = 2x$

and by the first result established in (ii):

$y_2 = 2xy_1 - 2y_0 = 4x^2 - 2$

So $y_1^2 - y_0y_2 = 4x^2 - (4x^2 - 2) = 2 > 0$

Thus the result is true for $n = 1$.

Then, if true for $n = k$, the 2nd result established in (ii)

$\Rightarrow$ it is true for $n = k + 1$, as the expression for $n = k + 1$ equals $2n \times$ (the expression for $n = k$) $+ 2y_n^2$; ie the sum of a positive quantity and a non-negative quantity.

And so (by the same reasoning as before), the required result is proved by induction.