STEP 2018, P3, Q1 - Solution (4 pages; 22/5/19)

(i) $f(\beta) = \beta - \frac{1}{\beta} - \frac{1}{\beta^2}$

$f'(\beta) = 1 + \frac{1}{\beta^2} + \frac{2}{\beta^3}$

$f'(\beta) = 0 \Rightarrow \beta^3 + \beta + 2 = 0$

$f'(-1) = 0$, so there is a stationary point at $(-1, -1)$

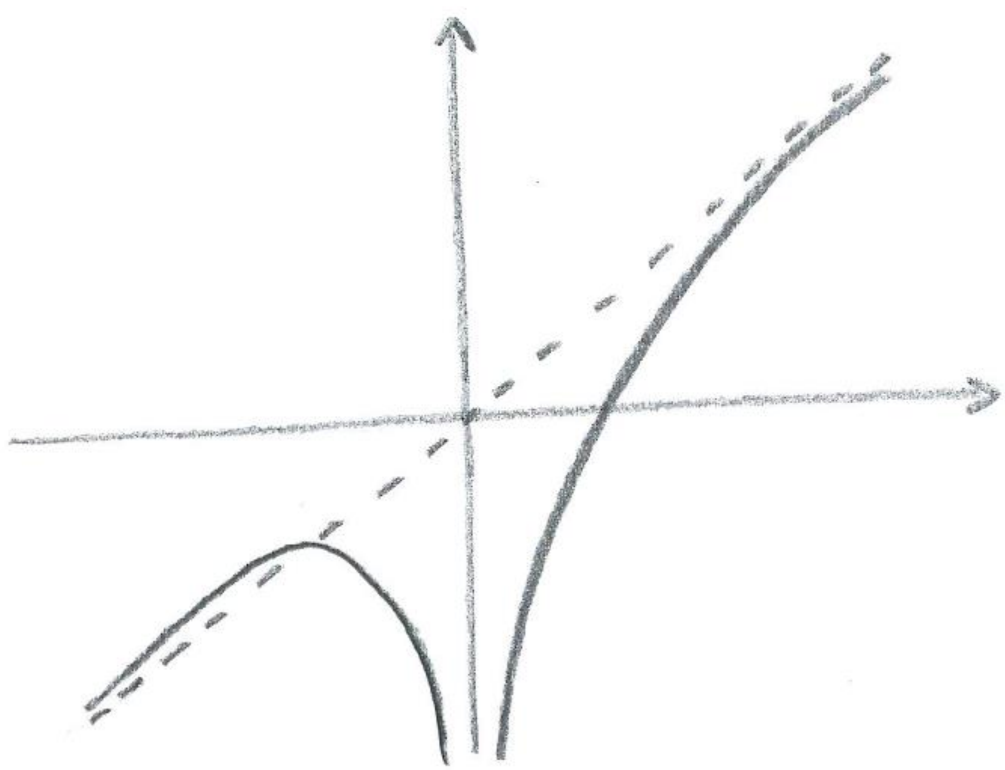
And, as $\frac{d}{d\beta}(\beta^3 + \beta + 2) = 3\beta^2 + 1 > 0$, the cubic $y = \beta^3 + \beta + 2$ crosses the β-axis only once, and so $(-1, -1)$ is the only stationary point.

Also, $f''(\beta) = -\frac{2}{\beta^3} - \frac{6}{\beta^4}$

and $f''(-1) = -4 < 0$, so that the point is a local maximum.

As $\beta \to 0^+$, $f(\beta) \to -\infty$, and as $\beta \to 0^-$, $f(\beta) \to -\infty$ also.

Also, as $\beta \to \infty$, $f(\beta) \to \beta^-$, whilst as $\beta \to -\infty$, $f(\beta) \to \beta^+$



$g(\beta) = \beta + \frac{3}{\beta} - \frac{1}{\beta^2}$

$g'(\beta) = 1 - \frac{3}{\beta^2} + \frac{2}{\beta^3}$

$g'(\beta) = 0 \Rightarrow \beta^3 - 3\beta + 2 = 0$

$\Rightarrow (\beta - 1)(\beta^2 + \beta - 2) = 0$

$\Rightarrow (\beta - 1)(\beta + 2)(\beta - 1) = 0$

So there are stationary points at $(1,3)$ *and* $(-2, -\frac{15}{4})$

$g''(\beta) = \frac{6}{\beta^3} - \frac{6}{\beta^4}$, and $g''(1) = 0$ *and* $g''(-2) < 0$

[The fact that 1 is a repeated root of $g'(\beta) = 0$ also means that

$g''(1) = 0$]

So $(-2, -\frac{15}{4})$ is a local maximum.

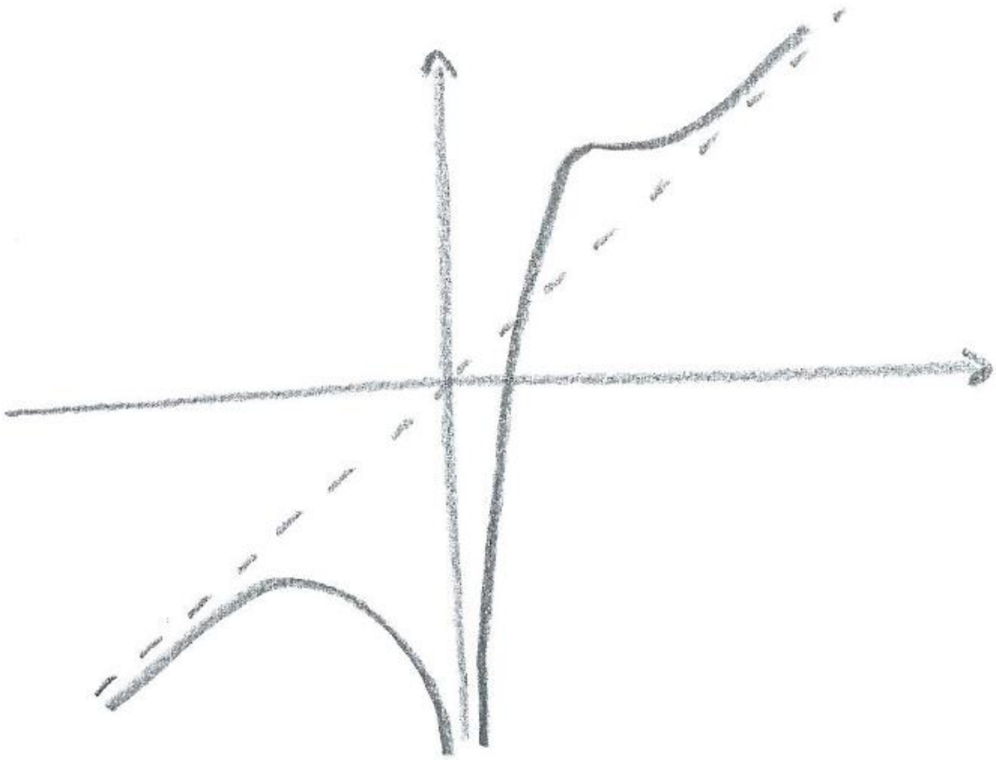
$g'''(\beta) = -\frac{18}{\beta^4} + \frac{24}{\beta^5}$, and $g'''(1) \neq 0$

So, as the 1st non-vanishing derivative (after the 1st one) is an odd one, $(1,3)$ is a point of inflexion [a turning point of the gradient, where $g''(\beta)$ changes sign].

As $\beta \to 0^+$, $g(\beta) \to -\infty$, and as $\beta \to 0^-$, $g(\beta) \to -\infty$ also.

[$-\frac{1}{\beta^2}$ is the critical term, as for $f(\beta)$]

Also, as $\beta \to \infty$, $g(\beta) \to \beta^+$, whilst as $\beta \to -\infty$, $g(\beta) \to \beta^-$



(ii) $u + v + \frac{1}{uv} = -\alpha + \frac{1}{\beta}$

$\frac{1}{u} + \frac{1}{v} + uv = \frac{v+u}{uv} + uv = \frac{-\alpha}{\beta} + \beta$

(iii) $u + v + \frac{1}{uv} = -1 \Rightarrow -\alpha + \frac{1}{\beta} = -1 \Rightarrow \alpha = \frac{1}{\beta} + 1$

[We may be able to use the fact that $f(\beta) \leq -1$ for $\beta < 0$; so we need to eliminate α.]

and $\frac{1}{u} + \frac{1}{v} + uv = \frac{-\alpha}{\beta} + \beta = -\frac{1}{\beta}\left(\frac{1}{\beta} + 1\right) + \beta = f(\beta)$

Then, from the sketch in (i), if $\beta < 0, \frac{1}{u} + \frac{1}{v} + uv \leq -1$

Real roots of quadratic $\Rightarrow \alpha^2 - 4\beta \geq 0 \Leftrightarrow \left(\frac{1}{\beta} + 1\right)^2 - 4\beta \geq 0$ (1)

If $\beta > 0$, then (1) $\Leftrightarrow (1 + \beta)^2 - 4\beta^3 \geq 0$

$\Leftrightarrow 1 + 2\beta + \beta^2 - 4\beta^3 \geq 0$ (2)

[Note: $f(1) = -1$, so $\beta = 1$ is likely to be a critical value.]

As the LHS $= 0$ when $\beta = 1$,

(2) $\Leftrightarrow (\beta - 1)(-4\beta^2 - 3\beta - 1) \geq 0$

$\Leftrightarrow (\beta - 1)(4\beta^2 + 3\beta + 1) \leq 0$ (3)

Then, as $4\beta^2 + 3\beta + 1 = 4(\beta + \frac{3}{8})^2 - \frac{9}{16} + 1 > 0$,

(3) $\Leftrightarrow \beta \leq 1$, so that $f(\beta) \leq f(1) = -1$

Thus, for all possible values of β, $\frac{1}{u} + \frac{1}{v} + uv \leq -1$, as required.

(iv) $u + v + \frac{1}{uv} = 3 \Rightarrow -\alpha + \frac{1}{\beta} = 3 \Rightarrow \alpha = \frac{1}{\beta} - 3$

and $\frac{1}{u} + \frac{1}{v} + uv = \frac{-\alpha}{\beta} + \beta = -\frac{1}{\beta}\left(\frac{1}{\beta} - 3\right) + \beta = g(\beta)$

Real roots of quadratic $\Rightarrow \alpha^2 - 4\beta \geq 0 \Leftrightarrow \left(\frac{1}{\beta} - 3\right)^2 - 4\beta \geq 0$

$\Leftrightarrow (1 - 3\beta)^2 - 4\beta^3 \geq 0$

$\Leftrightarrow 4\beta^3 - 9\beta^2 + 6\beta - 1 \leq 0$ (4)

As the LHS $= 0$ when $\beta = 1$,

(4) $\Leftrightarrow (\beta - 1)(4\beta^2 - 5\beta + 1) \leq 0$

$\Leftrightarrow (\beta - 1)(4\beta - 1)(\beta - 1) \leq 0$

$\Leftrightarrow 4\beta - 1 \leq 0$; ie $\beta \leq \frac{1}{4}$, or $\beta = 1$

So the greatest value of $\frac{1}{u} + \frac{1}{v} + uv$ is $g(1) = 3$.