STEP 2018, P2, Q8 - Solution (2 pages; 6/10/18)

(i) $v = y^{\frac{1}{2}} \Rightarrow y = v^2 \Rightarrow \frac{dy}{dt} = 2v\frac{dv}{dt}$

Then $\frac{dy}{dt} = \alpha y^{\frac{1}{2}} - \beta y \Rightarrow 2v\frac{dv}{dt} = \alpha v - \beta v^2$

$\Rightarrow v = 0$ or $2\int \frac{1}{\alpha-\beta v}\, dv = \int dt$,

so that either $y = 0$ or $t + C = -\frac{2}{\beta}\ln|\alpha - \beta v|$

$\Rightarrow \alpha - \beta v = exp\{-\frac{\beta}{2}(t + C)\}$

$\Rightarrow v = \frac{1}{\beta}[\alpha - exp\left\{-\frac{\beta}{2}(t + C)\right\}]$

So $y = 0$ *or* $y(t) = \frac{1}{\beta^2}[\alpha - exp\left\{-\frac{\beta}{2}(t + C)\right\}]^2$

[It isn't clear why the independent variable now changes from t to x - perhaps it's a mistake!]

$t = 0, y = 0 \Rightarrow \alpha - exp\left\{-\frac{\beta}{2}(t + C)\right\} = 0$

$\Rightarrow ln\alpha = -\frac{\beta C}{2} \Rightarrow C = -\frac{2}{\beta}ln\alpha$

So $y_1(x) = \frac{1}{\beta^2}[\alpha - exp\left\{-\frac{\beta}{2}\left(x - \frac{2}{\beta}ln\alpha\right)\right\}]^2$

$= \frac{1}{\beta^2}[\alpha - \alpha exp\left\{-\frac{\beta}{2}x\right\}]^2$

$= \frac{\alpha^2}{\beta^2}[1 - exp\left\{-\frac{\beta}{2}x\right\}]^2$

(ii) [If in doubt, try the simplest possible approach]

$v = y^{\frac{1}{3}} \Rightarrow y = v^3 \Rightarrow \frac{dy}{dt} = 3v^2\frac{dv}{dt}$

Then $\frac{dy}{dt} = \alpha y^{\frac{2}{3}} - \beta y \Rightarrow 3v^2 \frac{dv}{dt} = \alpha v^2 - \beta v^3$

$\Rightarrow v = 0$ or $3 \int \frac{1}{\alpha - \beta v}\, dv = \int dt$

ie as for (i), but with a 3 instead of a 2

So $y = 0$ or $y(t) = \frac{1}{\beta^2}[\alpha - exp\left\{-\frac{\beta}{3}(t + C)\right\}]^2$

and $y_2(x) = \frac{\alpha^2}{\beta^2}[1 - exp\left\{-\frac{\beta}{3}x\right\}]^2$

(iii) When $\alpha = \beta$, $y_1(x) = [1 - exp\left\{-\frac{\beta}{2}x\right\}]^2$

and $y_2(x) = [1 - exp\left\{-\frac{\beta}{3}x\right\}]^2$

As $x \to -\infty$, $y_1(x) \to \infty$ and as $x \to \infty$, $y_1(x) \to 1$

And $y_2(x)$ is obtained from $y_1(x)$ by replacing x with $\frac{2x}{3}$; ie by applying a stretch of scale factor $\frac{3}{2}$ in the x direction.