STEP 2018, P2, Q4 - Solution (2 pages; 17/11/18)

(i) [It is likely that values of n or p beyond a certain point can be ruled out.]

If $n \geq 5$, then $5|n! + 5$, and so $n! + 5$ cannot be prime.

$n = 1 \Rightarrow 6 = p$; so no sol'n

$n = 2 \Rightarrow 7 = p$; a sol'n

$n = 3 \Rightarrow 11 = p$; a sol'n

$n = 4 \Rightarrow 29 = p$; a sol'n

So sol'ns are (2,7), (3,11)& (4,29).

(ii) If $n \geq 7$, theorem 1$\Rightarrow m > 4n$

$\Rightarrow LHS = 3!\,5!\,\ldots(2n-1)!$ and $RHS = (4n)!\,(4n+1)\ldots$

But theorem 2 $\Rightarrow$ there is a prime number, p between $2n$ & $4n$

But then p is a factor of the RHS, but not of the LHS.

Hence $n < 7$.

$n = 1 \Rightarrow m = 1$

$n = 2 \Rightarrow m = 3$

$n = 3 \Rightarrow 6 \times 120 = m! \Rightarrow m = 6$

$n = 4 \Rightarrow m! = 6 \times 120 \times 7!$

$= 7! \times 8 \times (3 \times 30) = 8! \times 9 \times 10 = 10!$

For $n = 5$, LHS $= 3!\,5!\,7!\,9! \Rightarrow m < 11$, as $11 \nmid LHS$, but $11|m!$ when $m \geq 11$;

but $m = 10 \Rightarrow n = 4$, and $m < 10 \Rightarrow n < 4$

so $n \neq 5$

For $n = 6$, LHS $= 3!\,5!\,7!\,9!\,11! \Rightarrow m < 13$, as $13 \nmid LHS$, but $13|m!$

when $m \geq 13$;

as before, $m \leq 10 \Rightarrow n \leq 4$;

$m = 11$ isn't possible, as $LHS > 11!$

$m = 12$ isn't possible, as $LHS > 12!$

so $n \neq 6$

Hence sol'ns are (1,1), (2,3), (3,6), (4,10).