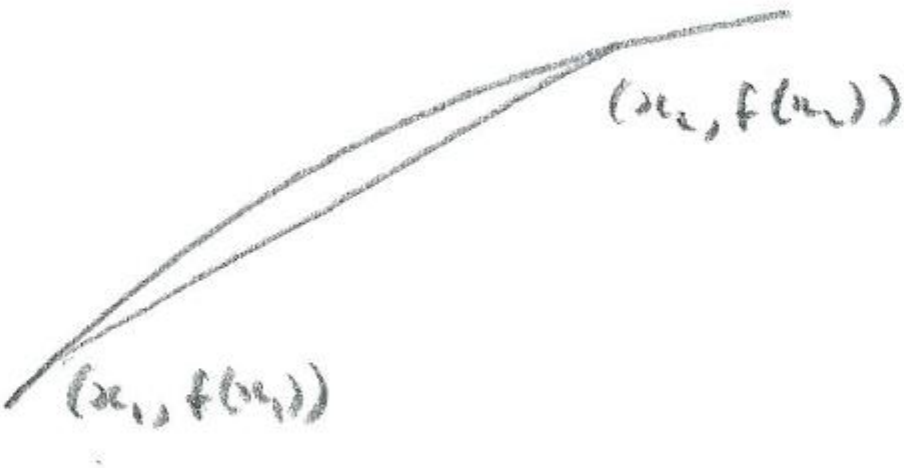
STEP 2018, P2, Q2 - Solution (2 pages; 29/10/18)



$f''(x) < 0 \ \ for\ a < x < b \Leftrightarrow f'(x)$ is decreasing for $a < x < b$, as in the diagram

(i) We can try splitting up $\frac{u+v+w}{3}$ as $(\frac{u}{3} + \frac{v}{6}) + (\frac{v}{6} + \frac{w}{3})$, initially.

[This involves only a minimal change (anything more complicated couldn't be relied on to work), and produces two terms of the same form.]

We can then write this as $\frac{1}{2}\left(\frac{2}{3}u + \frac{1}{3}v\right) + \frac{1}{2}\left(\frac{1}{3}v + \frac{2}{3}w\right)$, enabling the inequality in the question to be applied, to give

$f\left(\frac{u+v+w}{3}\right) \geq \frac{1}{2}f\left(\frac{2}{3}u + \frac{1}{3}v\right) + \frac{1}{2}f\left(\frac{1}{3}v + \frac{2}{3}w\right)$

$\geq \frac{1}{2}\left\{\frac{2}{3}f(u) + \frac{1}{3}f(v)\right\} + \frac{1}{2}\left\{\frac{1}{3}f(v) + \frac{2}{3}f(w)\right\}$

$= \frac{1}{3}\{f(u) + f(v) + f(w)\}$, as required.

(ii) Let $f(x) = sinx$, with $a = 0, b = \pi$; noting that $f(x)$ is concave.

Then, from (i),

$\frac{1}{3}\{sinA + sinB + sinC\} \leq \sin\left(\frac{A+B+C}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

so that $sinA + sinB + sinC \leq \frac{3\sqrt{3}}{2}$, as required.

(iii) $sinA \times sinB \times sinC \leq \frac{3\sqrt{3}}{8}$

$\Leftrightarrow ln(sinA) + ln(sinB) + ln(sinC) \leq ln(\frac{3\sqrt{3}}{8})$ (*)

(as $y = lnx$ is an increasing function) [ie $a \leq b \Leftrightarrow lna \leq lnb$]

Now setting $f(x) = ln(sinx)$, with $0 < x < \pi$,

$f'(x) = \frac{1}{sinx}cosx$

and $f''(x) = \frac{-sin^2x - cos^2x}{sin^2x} = -cosec^2x < 0$,

so that $f(x)$ is concave.

Then, from (i),

$ln(sinA) + ln(sinB) + ln(sinC) \leq 3\,ln\left(sin\left(\frac{A+B+C}{3}\right)\right)$

$= 3ln\left(sin\left(\frac{\pi}{3}\right)\right) = 3\,ln\left(\frac{\sqrt{3}}{2}\right) = ln\left(\frac{\sqrt{3}}{2}\right)^3 = l\,n\left(\frac{3\sqrt{3}}{8}\right)$,

establishing the equivalent result (*).