STEP 2018, P2, Q10 - Solution (3 pages; 29/9/18)

[Although this question involves an elastic string, there is no reference to the modulus of elasticity or stiffness, and so Hooke's law is not in fact needed. One unusual feature of the question is that we are asked to determine the speed of a point on the string (defined by its distance from the peg), rather than the speed of a given particle. However we can still use differentiation from 1st principles.]

Suppose that the given point moves a distance δx in time δt.

The required speed $\frac{dx}{dt} = \lim_{\delta t \to 0} \frac{\delta x}{\delta t}$

The length of the string at time t is $a + ut$, and at time $t + \delta t$ it is $a + u(t + \delta t)$.

The distance moved by a point on the string in time δt will be proportional to its distance from the peg (with a point at the peg moving no distance at all).

So $\delta x = u\delta t(\frac{x}{a+ut})$ and so $\frac{dx}{dt} = \lim_{\delta t \to 0} u\left(\frac{x}{a+ut}\right) = \frac{ux}{a+ut}$ (as this is a constant, for a given value of x and t).

Let x now be the distance of the ant from the peg at time t.

Then $\frac{dx}{dt} = \frac{ux}{a+ut} + v$

and $\frac{d}{dt}\left(\frac{x}{a+ut}\right) = \frac{(a+ut)\frac{dx}{dt} - xu}{(a+ut)^2} = \frac{\left(\frac{ux}{a+ut} + v\right)}{a+ut} - \frac{xu}{(a+ut)^2}$

$= \frac{v}{a+ut}$, as required.

Then $\frac{x}{a+ut} = \int \frac{v}{a+ut} dt = v\left(\frac{1}{u}\right) ln(a + ut) + C$

$t = 0, x = 0 \Rightarrow 0 = \frac{v}{u} lna + C,$

so that $\frac{x}{a+ut} = \frac{v}{u}\ln(\frac{a+ut}{a})$

At $t = T, x = a + uT$, so that $1 = \frac{v}{u}\ln(\frac{a+uT}{a})$

$\Rightarrow e^k = \frac{a+uT}{a}$, where $k = \frac{u}{v}$

and $uT = a(e^k - 1)$, as required.

[There is a 'lateral thinking' method that can be used for the last part, which becomes apparent when the answer is found by the following method.]

For the return journey, $\frac{dx}{dt} = \frac{ux}{a+ut} - v$ [note that the positive direction for x is still from left to right]

As the only change is that v has been replaced by $-v$, we can straightaway write that $\frac{d}{dt}\left(\frac{x}{a+ut}\right) = -\frac{v}{a+ut}$

$\Rightarrow \frac{x}{a+ut} = -\int \frac{v}{a+ut}dt = -v\left(\frac{1}{u}\right) ln(a+ut) + C'$

$t = T, x = a + uT \Rightarrow 1 = C' - \frac{v}{u}\ln(a + uT)$

So $\frac{x}{a+ut} = 1 + \frac{v}{u}\ln(\frac{a+uT}{a+ut})$

Then $t = T + T', x = 0$ (where T' is the time for the return journey) $\Rightarrow 0 = 1 + \frac{v}{u}\ln\left(\frac{a+uT}{a+uT+uT'}\right)$

$\Rightarrow e^{-k} = \left(\frac{a+uT}{a+uT+uT'}\right)$

$\Rightarrow e^k = 1 + \frac{uT'}{a+uT}$

$\Rightarrow T' = \frac{1}{u}(e^k - 1)(a + uT)$

[This can also be deduced from the fact that the required time is the time that would be taken by another ant to walk from the peg to the end of the string (with speed v relative to the string), where the string has an initial length of $a + uT$.]