STEP 2018, P1, Q10 - Solution (3 pages; 14/5/20)

[The situation isn't that realistic, as there are no resistances for the engines. Obviously the driving forces can be considered to be net of resistances though.]

(i) Applying N2L to the leading engine:

$D - T = Ma$, where a is the acceleration

Applying N2L to the whole of the train:

$2D - nR = (2M + nm)a$

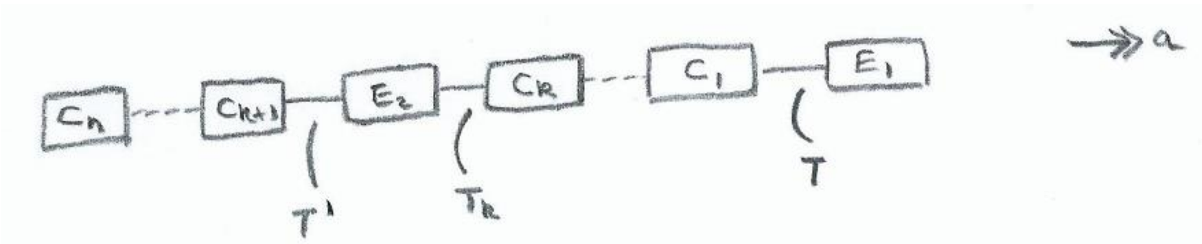
Then $a = \frac{D-T}{M} = \frac{2D-nR}{(2M+nm)}$ (1)

$\Rightarrow (D - T)(2M + nm) = M(2D - nR)$

$\Rightarrow -T(2M + nm) = 2MD - MnR - 2DM - Dnm$

$\Rightarrow T = \frac{-MnR-Dnm}{-(2M+nm)} = \frac{n(mD+MR)}{nm+2M}$, as required

(ii) [Note that the carriages are counted from the front of the train - which may be on the right, if a diagram is drawn.]



Referring to the diagram, let T_k be the tension to the left of the kth carriage, and T' the tension to the left of the 2nd engine (the one not at the front).

Then, applying N2L to C_1:

$T - T_1 - R = ma$, so that $T_1 = T - R - ma < T$

Similarly, applying N2L to C_2:

$T_1 - T_2 - R = ma$, so that $T_2 = T_1 - R - ma < T_1$

So T is greater than $T_1, T_2, \ldots, T_k$

Also, applying N2L to C_{k+1}:

$T' - T_{k+1} - R = ma$, so that $T_{k+1} = T' - R - ma < T'$

Similarly, applying N2L to C_{k+2}:

$T_{k+1} - T_{k+2} - R = ma$, so that $T_{k+2} = T_{k+1} - R - ma < T_{k+1}$

So T' is greater than $T_{k+1}, T_{k+2}, \ldots, T_n$

Hence, T will be greater than all the other tensions if $T > T'$.

Applying N2L to all the carriages behind E_2 (ie $C_{k+1} + \cdots + C_n$):

$T' - (n-k)R = (n-k)ma$

$\Rightarrow T' = (n-k)\{R + m\left(\frac{2D-nR}{2M+nm}\right)\}$, from (1) in (i)

$= \frac{(n-k)}{(2M+nm)}\{R(2M+nm) + 2mD - mnR\}$

$= \frac{(n-k)}{(2M+nm)}\{2RM + 2mD\}$

From (i), $T = \frac{n(mD+MR)}{nm+2M}$,

so that $T > T'$ provided that $n > 2(n-k)$;

ie $2k > n$, or $k > \frac{n}{2}$

(iii) Applying N2L to C_n: $T_{n-1} - R = ma$,

so that $T_{n-1} = R + ma > 0$

And ... $T_{k+2} < T_{k+1} < T'$, from (ii).

Thus, all of the couplings to the left of E_2 will be in tension.

Also, from (i), $T > T_1 > T_2 > \cdots > T_k$

So we need only consider T_k (if T_k isn't a compression, none of the others will be).

Applying N2L to $(C_n + \cdots + C_{k+1} + E_2)$:

$T_k + D - (n-k)R = [(n-k)m + M]a$,

so that $T_k = [(n-k)m + M]\frac{(2D-nR)}{(2M+nm)} - D + (n-k)R$

$= \frac{1}{(2M+nm)}\{2(n-k)mD - (n-k)mnR + 2MD - MnR$

$+(n-k)R(2M+nm) - D(2M+nm)\}$

To show that $T_k < 0$, consider the denominator:

$2(n-k)RM - Dnm + 2nmD - 2kmD - MnR$

$= MnR - 2kRM + Dnm - 2kmD$

$= MR(n-2k) + Dm(n-2k)$

$= (MR+Dm)(n-2k) < 0$, provided that $k > \frac{n}{2}$