STEP 2018, Paper 1, Q2 Solution (3 pages; 22/5/25)

2 If $x = \log_b c$, express c in terms of b and x and prove that $\dfrac{\log_a c}{\log_a b} = \log_b c$.

(i) Given that $\pi^2 < 10$, prove that

$$\frac{1}{\log_2 \pi} + \frac{1}{\log_5 \pi} > 2\,.$$

(ii) Given that $\log_2 \dfrac{\pi}{\mathrm{e}} > \dfrac{1}{5}$ and that $\mathrm{e}^2 < 8$, prove that $\ln \pi > \frac{17}{15}$.

(iii) Given that $\mathrm{e}^3 > 20$, $\pi^2 < 10$ and $\log_{10} 2 > \frac{3}{10}$, prove that $\ln \pi < \frac{15}{13}$.

Solution

1st Part

$x = log_b c \Leftrightarrow c = b^x$

2nd Part

Write $y = log_a b$ and $z = log_a c$,

so that $b = a^y$ and $c = a^z$

[result to prove : $\frac{z}{y} = x$]

Then $b^x = (a^y)^x = a^{yx}$

Thus $c = b^x = a^{yx}$ and also $c = a^z$,

so that $z = yx$, and hence $\frac{z}{y} = x$;

ie $\frac{log_a c}{log_a b} = log_b c$, as required.

(i) $\pi^2 < 10 \Rightarrow log_\pi 10 > 2$ (*)

Also, from the 2nd Part with $a = c$, $\frac{1}{log_c b} = log_b c$

So $\frac{1}{log_2 \pi} + \frac{1}{log_5 \pi} = log_\pi 2 + log_\pi 5 = log_\pi (2 \times 5) > 2$, from (*),

as required.

(ii) $e^2 < 8 \Rightarrow log_e 8 > 2$ (*)

From the initial 2nd Part, $ln\pi = log_e \pi = \frac{log_a \pi}{log_a e}$

[The fact that $log_e 8 = 3log_e 2$ suggests setting $a = 2$, as we are told that $log_2\left(\frac{\pi}{e}\right) > \frac{1}{5}$]

$= \frac{log_2 \pi}{log_2 e}$, setting a equal to 2.

Now $\frac{1}{log_2 e} = log_e 2$ (=setting $a = c = 2$ & $b = e$ in the initial 2nd Part), and $3log_e 2 = log_e 8 \ > 2$, from (*),

so that $\frac{1}{log_2 e} = log_e 2 > \frac{2}{3}$

Also, we are told that $log_2 \left(\frac{\pi}{e}\right) > \frac{1}{5}$, so that $log_2 \pi - log_2 e > \frac{1}{5}$,

and hence $\ln\pi = \frac{log_2 \pi}{log_2 e} = \frac{log_2 \pi - log_2 e}{log_2 e} + \frac{log_2 e}{log_2 e} > \frac{1}{5} \cdot \frac{2}{3} + 1 = \frac{17}{15}$, as required.

(iii) $e^3 > 20 \Rightarrow log_e 20 < 3$, (**)

and $\pi^2 < 10 \Rightarrow log_\pi 10 > 2$ (***)

From the initial 2nd Part again, $ln\pi = log_e \pi = \frac{log_{10}\pi}{log_{10} e}$

And $log_{10}\pi = \frac{1}{log_\pi 10} < \frac{1}{2}$, from (***).

Also $log_e 20 = log_e 2 + log_e 10 = \frac{log_{10} 2}{log_{10} e} + \frac{1}{log_{10} e}$, from the initial 2nd Part once again.

And so, from (**), $\frac{log_{10} 2}{log_{10} e} + \frac{1}{log_{10} e} < 3$,

and hence $\frac{1}{log_{10} e} < \frac{3}{(log_{10} 2 + 1)} < \frac{3}{(\frac{3}{10} + 1)} = \frac{30}{13}$

(as we are told that $log_{10} 2 > \frac{3}{10}$)

Then $ln\pi = \frac{log_{10}\pi}{log_{10} e} < \frac{1}{2} \cdot \frac{30}{13} = \frac{15}{13}$, as required.