STEP 2017, P3, Q3 - Solution (2 pages; 13/7/20)

1st part

$-A = (\alpha\beta + \gamma\delta) + (\alpha\gamma + \beta\delta) + (\alpha\delta + \beta\gamma) = q,$

so that $A = -q$

(i) Let $f(y) = y^3 - 3y^2 - 40y + (120 - 36)$

Then $f(2) = 8 - 12 - 80 + 84 = 0$, so that $y - 2$ is a factor of $f(y)$, and $f(y) = (y-2)(y^2 - y - 42) = (y-2)(y+6)(y-7)$

So, $\alpha\beta + \gamma\delta$ (the largest root of $f(y) = 0$) is 7.

(ii) **1st part**

[We can try using the value of q for the 1st result, and the value of s for the 2nd result.]

$(\alpha + \beta)(\gamma + \delta) = \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta$

$= \{\sum r_i\} - (\alpha\beta + \gamma\delta)$, where r_i are the 3 roots of the cubic

$= q - 7$,

as $\sum r_i$ is the sum of the roots of the quartic, taken 2 at a time

$= 3 - 7 = -4$

2nd part

$\alpha\beta + \gamma\delta = 7$ & $10 = s = (\alpha\beta)(\gamma\delta)$ (from the quartic)

and so $\alpha\beta = 5$ & $\gamma\delta = 2$ (given that $\alpha\beta > \gamma\delta$)

(iii) $p = 0 \Rightarrow \alpha + \beta + \gamma + \delta$ (1) (from the quartic)

and $r = -6 \Rightarrow \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = 6$

(also from the quartic)

$\Rightarrow \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = 6$ (2)

Let $X = \alpha + \beta$, so that $\gamma + \delta = -X$ (from (1))

Then, as $\alpha\beta = 5$ & $\gamma\delta = 2$,

(2) $\Rightarrow 5(-X) + 2X = 6$, so that $X = -2$

So $\alpha + \beta = -2$ & $\alpha\beta = 5$,

and $\gamma + \delta = 2$ & $\gamma\delta = 2$

Then α & β are the roots of $x^2 + 2x + 5 = 0$,

giving $x = \frac{-2\pm\sqrt{-16}}{2} = -1 \pm 2i$

And γ & δ are the roots of $y^2 - 2y + 2 = 0$,

giving $y = \frac{2\pm\sqrt{-4}}{2} = 1 \pm i$

Thus the roots of the quartic are $-1 \pm 2i$ & $1 \pm i$

Notes

(i) This last part hasn't in fact used the result $(\alpha + \beta)(\gamma + \delta) = -4$ (though this can be used as a check). However, the official mark scheme doesn't even mention p & r in part (iii).

(ii) Note that the quartic has complex roots, even though the variable is x (rather than z) - presumably so as not to give the game away.