STEP 2017, P3, Q12 - Solution (2 pages; 14/7/20)

(i) **1st part**

$P(X = x) = \sum_{y=1}^{n} k(x + y)$

$= kxn + k.\frac{1}{2}n(n + 1)$

And $\sum_{x=1}^{n} \sum_{y=1}^{n} k(x + y) = 1$,

so that $2\{kn.\frac{1}{2}n(n + 1)\} = 1$ (by symmetry),

and hence $k = \frac{1}{n^2(n+1)}$

So $P(X = x) = \frac{1}{n^2(n+1)}(xn + \frac{1}{2}n(n + 1))$

$= \frac{2x+n+1}{2n(n+1)}$ or $\frac{n+1+2x}{2n(n+1)}$, as required.

2nd part

$P(X = x|Y = y) = \frac{P(X=x,Y=y)}{P(Y=y)}$

By symmetry, $P(Y = y) = \frac{n+1+2y}{2n(n+1)}$,

so that $\frac{P(X=x,Y=y)}{P(Y=y)} = \frac{k(x+y)}{\left(\frac{n+1+2y}{2n(n+1)}\right)}$

X and Y independent $\Leftrightarrow P(X = x|Y = y) = P(X = x)$ for all x & y

$\Leftrightarrow \frac{k(x+y)}{\left(\frac{n+1+2y}{2n(n+1)}\right)} = \frac{n+1+2x}{2n(n+1)}$

[Alternatively, $P(X = x|Y = y) = P(X = x)\, P(Y = y)$ can just be quoted as the condition for independence.]

$\Leftrightarrow k(x + y)\big(2n(n + 1)\big)^2 = (n + 1 + 2x)(n + 1 + 2y)$ (1)

and as $k = \frac{1}{n^2(n+1)}$; writing $N = n + 1$,

$(1) \Leftrightarrow 4(x + y)N = (N + 2x)(N + 2y)$

$\Leftrightarrow (N - 2x)(N - 2y) = 0$,

which is only true when $x = \frac{n+1}{2}$ or $y = \frac{n+1}{2}$

As it isn't true for all x & y, X and Y are not independent.

(ii) $Cov(X, Y) = E(XY) - E(X)E(Y)$

$E(XY) = \sum_{x=1}^{n} \sum_{y=1}^{n} k(x + y)xy$

$= 2k\{\frac{1}{2}n(n + 1) \sum_{x=1}^{n} x^2\}$, by symmetry

$= \frac{1}{n} . \frac{1}{6} n(n + 1)(2n + 1)$, as $k = \frac{1}{n^2(n+1)}$

$= \frac{1}{6}(n + 1)(2n + 1)$

And $E(X) = \sum_{x=1}^{n} P(X = x)x$

$= \sum_{x=1}^{n} \frac{n+1+2x}{2n(n+1)} . x$

$= \frac{1}{2n} . \frac{1}{2} n(n + 1) + \frac{1}{n(n+1)} . \frac{1}{6} n(n + 1)(2n + 1)$

$= \frac{1}{12}(3(n + 1) + 2(2n + 1))$

$= \frac{1}{12}(7n + 5)$

And, by symmetry, $E(Y) = \frac{1}{12}(7n + 5)$ also.

So $Cov(X, Y) = \frac{1}{6}(n + 1)(2n + 1) - \frac{1}{144}(7n + 5)^2$

$= \frac{1}{144}\{24(2n^2 + 3n + 1) - (49n^2 + 70n + 25)\}$

$= \frac{1}{144}\{-n^2 + 2n - 1\} = -\frac{1}{144}(n - 1)^2 < 0$, as $n \geq 2$; as required