STEP 2017, P2, Q10 - Solution (3 pages; 27/5/20)

1st part

Let $F(x)$ be the force applied by the engine.

Then, by N2L, $F(x) - Av^2 - R = ma$,

and the work done by the engine is $\int_0^d F(x)dx$

$= \int_0^d (ma + R + Av^2)\, dx$, as required.

2nd part

$\frac{dv}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = a \cdot \frac{1}{v}$, and so $dx = dv \cdot \frac{v}{a}$

When $x = 0, v = 0$,

and $'v^2 = u^2 + 2as' \Rightarrow v = \sqrt{2ad} = w$ when $x = d$

Then $\int_0^d (ma + R + Av^2)\, dx = \int_0^w \frac{(ma+R+Av^2)v}{a}\, dv$, as required.

(i) Work done for the 2nd half is $\int_w^{w_1} \frac{(-ma+R+Av^2)v}{-a}\, dv$,

where ${w_1}^2 = w^2 + 2(-a)d = 2ad - 2ad = 0$

So work done for 2nd half

$= [\frac{1}{2}\left(m - \frac{R}{a}\right) v^2 - \frac{Av^4}{4a}]_w^0$

$= \frac{1}{2}\left(m - \frac{R}{a}\right)(0 - w^2) - \frac{A}{4a}(0 - w^4)$

$= \frac{1}{2}\left(m - \frac{R}{a}\right)(-2ad) + \frac{A}{4a}(2ad)^2$

$= -mad + Rd + Aad^2$

Work done for the 1st half is $[\frac{1}{2}\left(m+\frac{R}{a}\right)v^2+\frac{Av^4}{4a}]_0^w$

$=\frac{1}{2a}(ma+R)(2ad)+\frac{A(2ad)^2}{4a}$

$=(ma+R)d+Aad^2$

So total work done is

$(-mad+Rd+Aad^2)+(ma+R)d+Aad^2$

$=2Aad^2+2Rd$, as required.

Note that, for the 1st half of the journey,

$F(x)=ma+R+Av^2>0$ (as A & R must be positive)

And for the 2nd half of the journey,

$F(x)=-ma+R+Av^2>0,$ as $R>ma$

So the engine is doing positive work at all times.

(ii) $F(x)$ falls to zero (during the 2nd half of the journey) when

$-ma+R+Av^2=0;$

ie when $Av^2=ma-R$, and $v=\sqrt{\frac{ma-R}{A}}$ (which is defined,

as $R<ma$)

In order for the speed to equal $\sqrt{\frac{ma-R}{A}}$ at some point,

$\sqrt{\frac{ma-R}{A}}<w=\sqrt{2ad},$

so that $\frac{ma-R}{A}<2ad;\ ma-R<2Aad$, and $R>ma-2Aad$

(as given).

Let $w' = \sqrt{\frac{ma-R}{A}}$

Then work done for the whole journey

$= (ma + R)d + Aad^2$ [from (i)]

$+[\frac{1}{2}\left(m - \frac{R}{a}\right)v^2 - \frac{Av^4}{4a}]_w^{w'}$ [also from (i)]

$= (ma + R)d + Aad^2$

$+\{\frac{1}{2}\left(m - \frac{R}{a}\right)\left(\frac{ma-R}{A}\right) - \frac{A}{4a}\left(\frac{ma-R}{A}\right)^2 - \frac{1}{2}\left(m - \frac{R}{a}\right)(2ad) + \frac{A}{4a}(2ad)^2\}$

$= (ma + R)d + Aad^2 + \frac{(ma-R)^2}{Aa}\left(\frac{1}{2} - \frac{1}{4}\right) - mad + Rd + Aad^2$

$= 2Rd + 2Aad^2 + \frac{(ma-R)^2}{4Aa}$, as required.