STEP 2016, Paper 2, Q6 – Solution (2 pages; 8/6/18)

(i) $(1-x^2)\left(\frac{dy}{dx}\right)^2 + y^2 = 1$ (1)

$y = x \Rightarrow LHS\ of\ (1) = (1-x^2)(1)^2 + x^2 = 1 = RHS$

Also $y(1) = 1$.

ie $y = x$ satisfies DE and boundary condition, so that $y_1(x) = x$

(ii) $(1-x^2)\left(\frac{dy}{dx}\right)^2 + 4y^2 = 4$ (2)

$y = 2x^2 - 1 \Rightarrow$

$LHS\ of\ (2) = (1-x^2)(4x)^2 + 4(2x^2-1)^2$

$= 16x^2 - 16x^4 + 16x^4 - 16x^2 + 4 = 4 = RHS$

Also $y(1) = 1$.

ie $y = 2x^2 - 1$ satisfies DE and boundary condition, so that
$y_2(x) = 2x^2 - 1$

(iii) $\frac{dz}{dx} = 4y_n(x)\frac{d}{dx}\left(y_n(x)\right)$,

so that $(1-x^2)\left(\frac{dz}{dx}\right)^2 + 4n^2z^2$

$= (1-x^2).16\left(y_n(x)\right)^2\left(\frac{d}{dx}y_n(x)\right)^2 + 4n^2[2(y_n(x))^2 - 1]^2$ (3)

Also $(1-x^2)\left(\frac{d}{dx}[y_n(x)]\right)^2 + n^2(y_n(x))^2 = n^2$,

so that (3) $= 16(y_n(x))^2[n^2 - n^2\left(y_n(x)\right)^2]$

$+4n^2\left[4\left(y_n(x)\right)^4 + 1 - 4\left(y_n(x)\right)^2\right] = 4n^2$, as required.

Also, $z(1) = 2(1)^2 - 1 = 1$,

so that $z(x) = y_{2n}(x)$,

and hence $y_{2n}(x) = 2(y_n(x))^2 - 1$

(iv) rtp (result to prove): $(1 - x^2)\left(\frac{dv}{dx}\right)^2 + (mn)^2 v^2 = (mn)^2$ (4),

and also that $v(1) = 1$

First of all, $v(1) = y_n\big(y_m(1)\big) = y_n(1) = 1$

Then $\frac{dv}{dx} = \frac{dy_n}{dy_m}\frac{dy_m}{dx}$ (5)

Also $y_n\big(y_m(x)\big)$ satisfies

$[1 - [y_m(x)]^2]\left(\frac{dy_n}{dy_m}\right)^2 + n^2[y_n\big(y_m(x)\big)]^2 = n^2$,

so that (from (5)),

$\left(\frac{dv}{dx}\right)^2 = \left(\frac{dy_n}{dy_m}\right)^2\left(\frac{dy_m}{dx}\right)^2 = \frac{n^2 - n^2 v^2}{1 - [y_m(x)]^2}\left(\frac{dy_m}{dx}\right)^2$ [as $v = y_n\big(y_m(x)\big)$]

and hence

$(1 - x^2)\left(\frac{dv}{dx}\right)^2 = (1 - x^2)\frac{n^2(1 - v^2)}{1 - [y_m(x)]^2}\left(\frac{dy_m}{dx}\right)^2$

And $(1 - x^2)\left(\frac{dy_m}{dx}\right)^2 + m^2(y_m(x))^2 = m^2$,

so that $(1 - x^2)\left(\frac{dv}{dx}\right)^2 = [m^2 - m^2\big(y_m(x)\big)^2]\frac{n^2(1 - v^2)}{1 - [y_m(x)]^2}$

$= m^2 n^2(1 - v^2)$, which gives (4).