STEP 2015, P3, Q6 - Solution (4 pages; 29/7/20)

(i) Eliminating z, $v = w^2 + (u-w)^2$,

so that $2w^2 - 2uw + u^2 - v = 0$

and $w = \frac{2u \pm \sqrt{4u^2 - 8(u^2 - v)}}{4} = \frac{u \pm \sqrt{2v - u^2}}{2}$

By symmetry, if $w = \frac{u + \sqrt{2v - u^2}}{2}$, then $z = \frac{u - \sqrt{2v - u^2}}{2}$ (and vice-versa).

Thus if u & v are real, and $u^2 \leq 2v$, then w & z will be real.

Now, $2v - u^2 = 2(w^2 + z^2) - (w^2 + z^2 + 2wz)$

$= w^2 + z^2 - 2wz$

$= (w - z)^2$

So if w & z are real, then u & v will be real (from their definitions), and $2v - u^2 = (w - z)^2 \geq 0$, so that $u^2 \leq 2v$.

Alternative method

$2v - u^2 = 2(w^2 + z^2) - (w^2 + z^2 + 2wz)$

$= w^2 + z^2 - 2wz$

$= (w - z)^2$

If w & z are real, then u & v will be real, and $2v - u^2 \geq 0$, so that

$u^2 \leq 2v$

Suppose now that u & v are real, and $u^2 \leq 2v$, so that $(w - z)^2$ is a non-negative real number.

Now, if $(re^{i\theta})^2 = r^2 e^{2\theta i}$ is a non-negative real number, then

$2\theta = n(2\pi)$, for some integer n, and so $\theta = n\pi$,

and therefore $re^{i\theta}$ is a real number.

So $w - z$ is a real number, a say.

Then, as $u = w + z$ is a real number, b say,

$w = \frac{1}{2}(a + b)$ & $z = \frac{1}{2}(b - a)$ are both real.

(ii) **1st part**

From the 1st eq'n, $u = w + z$

Let $v = w^2 + z^2$. Then, from the 2nd eq'n:

$2v - u^2 = 2(w^2 + z^2) - u^2 = 2\left(u^2 - \frac{2}{3}\right) - u^2$

$= u^2 - \frac{4}{3}$

So, if $u^2 < \frac{4}{3}$, then w & z will not be real (for the 2nd part).

Now $w^3 + z^3 = (w + z)(w^2 - wz + z^2)$

[Note that, if we write $f(w) = w^3 + z^3$, then $f(-z) = 0$, so that $w + z$ is a factor, by the Factor theorem.]

Then the 3rd eq'n becomes

$u\left(\left[u^2 - \frac{2}{3}\right] - wz\right) - \lambda u = -\lambda$ (A) , from the 1st & 2nd eq'ns

Also, $2wz = (w + z)^2 - (w^2 + z^2)$

$= u^2 - \left[u^2 - \frac{2}{3}\right]$, from the 1st & 2nd eq'ns

$= \frac{2}{3}$, so that $wz = \frac{1}{3}$

Then (A) becomes $u(u^2 - 1 - \lambda) + \lambda = 0$

or $f(u) = u^3 - (1+\lambda)u + \lambda = 0$

[See the official Hints & Sol'ns for a quicker method, using the fact that $f(u) = u(u^2-1) - \lambda(u-1)$, so that $u-1$ is a factor.]

In order for there to be 3 possible (real) values of u, we need to show that the minimum of $f(u)$ lies below the u-axis (for all positive values of λ except one).

$f'(u) = 3u^2 - (1+\lambda)$,

so that $f'(u) = 0 \Rightarrow u^2 = \frac{1+\lambda}{3}$

The minimum then occurs at the right-most value; ie $u = \sqrt{\frac{1+\lambda}{3}}$

and we require $f\left(\sqrt{\frac{1+\lambda}{3}}\right) < 0$ for all positive values of λ except one.

ie $\sqrt{\frac{1+\lambda}{3}}\left\{\left(\frac{1+\lambda}{3}\right) - (1+\lambda)\right\} + \lambda < 0$

$\Leftrightarrow -\frac{2}{3}(1+\lambda)\sqrt{\frac{1+\lambda}{3}} < -\lambda$

$\Leftrightarrow \frac{2}{3}(1+\lambda)\sqrt{\frac{1+\lambda}{3}} > \lambda$

$\Leftrightarrow \frac{4}{9}(1+\lambda)^2\left(\frac{1+\lambda}{3}\right) > \lambda^2$, as $a^2 > b^2 \Rightarrow a > b$ when $b > 0$

$\Leftrightarrow 4(1+\lambda)^3 > 27\lambda^2$

$\Leftrightarrow g(\lambda) = 4\lambda^3 - 15\lambda^2 + 12\lambda + 4 > 0$

As we are told that $g(\lambda) \leq 0$ for just one positive value of λ (say λ_0, it follows that λ_0 is a repeated root of $g(\lambda) = 0$,

and we can see that $g(2) = 32 - 60 + 24 + 4 = 0$

Then $g(\lambda) = (\lambda - 2)(4\lambda^2 - 7\lambda - 2) = (\lambda - 2)(\lambda - 2)(4\lambda + 1)$

Thus $g(\lambda)$ has a root at $\lambda = -\frac{1}{4}$ and a repeated root at $\lambda = 2$,

and therefore $g(\lambda) > 0$ for all positive values of λ except one; namely $\lambda = 2$.

2nd part

It was shown at the start of the 1st part that, if $u^2 < \frac{4}{3}$, then w & z will not be real.

[We just need to find one solution for the eq'ns such that w & z are not real.]

Consider $f(u) = u^3 - (1 + \lambda)u + \lambda = 0$ again,

and note that $f(1) = 0$ (for any λ in fact).

So suppose that $u = 1$ (so that $u^2 < \frac{4}{3}$).

Then $w + z = 1$, and $wz = \frac{1}{3}$, as before,

so that $w + \frac{1}{3w} = 1$, and hence $3w^2 - 3w + 1 = 0$,

giving $w = \frac{3 \pm \sqrt{9-12}}{6}$

By symmetry, if $w = \frac{3+i\sqrt{3}}{6}$, then $z = \frac{3-i\sqrt{3}}{6}$, which is the required counter-example.