STEP 2015, Paper 3, Q1 Solution (3 pages; 26/5/20)

(i) **1st part**

$I_n = \int_0^\infty \frac{1}{(1+u^2)^n}\ du$

$I_n - I_{n+1} = \int_0^\infty \frac{1}{(1+u^2)^n} - \frac{1}{(1+u^2)^{n+1}}\ du$

$= \int_0^\infty \frac{(1+u^2)-1}{(1+u^2)^{n+1}}\ du$

$= \int_0^\infty \frac{u^2}{(1+u^2)^{n+1}}\ du$

$= \frac{1}{2}\int_0^\infty u.\frac{2u}{(1+u^2)^{n+1}}\ du$

$=$ (by Parts) $\frac{1}{2}\left[u.\left(\frac{1}{-n}\right)\frac{1}{(1+u^2)^n}\right]_0^\infty - \frac{1}{2}\int_0^\infty \left(\frac{1}{-n}\right)\frac{1}{(1+u^2)^n}\ du$

$= 0 + \frac{1}{2n}\ I_n$, as required.

2nd part

Hence $I_{n+1} = I_n\left(1 - \frac{1}{2n}\right) = \frac{2n-1}{2n}I_n$

$= \frac{2n-1}{2n}\ .\frac{2n-3}{2n-2}\ .\frac{2n-5}{2n-4} \dots \frac{2(1)-1}{2(1)}\ I_1$

$= \frac{[\frac{(2n)!}{2^n n!}]}{2^n.n!}\int_0^\infty \frac{1}{1+u^2}\,du$

$= \frac{(2n)!}{(n!)^2(2^n)^2}\,[arctan\ u]_0^\infty$

$= \frac{(2n)!}{(n!)^2.2^{2n}}\,(\frac{\pi}{2})$

$= \frac{(2n)!\pi}{2^{2n+1}(n!)^2}$, as required.

(ii) $J = \int_0^\infty f((x - x^{-1})^2)dx$

Let $K = \int_0^\infty x^{-2} f((x - x^{-1})^2)dx$

and $L = \frac{1}{2}\int_0^\infty (1 + x^{-2}) f((x - x^{-1})^2)dx$

Let $u = x^{-1}$, so that $du = -x^{-2}dx$

Then $(x - x^{-1})^2 = (u^{-1} - u)^2 = (u - u^{-1})^2$, and the limits are reversed.

Then $K = \int_\infty^0 f((u - u^{-1})^2)(-du) = \int_0^\infty f((u - u^{-1})^2)du = J$

Also $L = \frac{1}{2}(J + K) = J$

Now let $u = x - x^{-1}$, so that $du = 1 + x^{-2}\ dx$

and $L = \frac{1}{2}\int_{-\infty}^\infty f(u^2)du$

Now, with the substitution $v = -u$,

$\int_{-\infty}^0 f(u^2)du = \int_\infty^0 f(v^2)(-dv) = \int_0^\infty f(v^2)dv,$

so that $L = \frac{1}{2}(\int_{-\infty}^0 f(u^2)du + \int_0^\infty f(u^2)du)$

$= \int_0^\infty f(u^2)du$

This establishes the required results.

(iii) $x^4 - x^2 + 1 = x^2(x^2 - 1 + x^{-2})$

And $(x - x^{-1})^2 = x^2 - 2 + x^{-2}$

So $\frac{x^{2n-2}}{(x^4 - x^2 + 1)^n} = x^{-2}.\frac{x^{2n}}{(x^2(x^2 - 1 + x^{-2}))^n}$

$= x^{-2}(x^2 - 1 + x^{-2})^{-n}$

$= x^{-2} f((x - x^{-1})^2),$

where $f(x) = (x+1)^{-n}$

Then $\int_0^\infty \frac{x^{2n-2}}{(x^4-x^2+1)^n}\, dx = \int_0^\infty x^{-2} f((x-x^{-1})^2)$

$= \int_0^\infty f(u^2) du$, from (ii)

$= \int_0^\infty (u^2+1)^{-n} du = I_n$

and from (i), $I_n = \frac{(2[n-1])!\pi}{2^{2[n-1]+1}([n-1]!)^2}$

$= \frac{(2n-2)!\pi}{2^{2n-1}([n-1]!)^2}$