STEP 2015, P1, Q12 - Solution (2 pages; 16/7/20)

(i) Number of casualties requiring surgery $\sim B(n, \frac{1}{4})$

So P(exactly r casualties require surgery) $= \binom{n}{r}\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{n-r}$

$= \frac{n!3^{n-r}}{r!(n-r)!4^n}$

(ii) Let Y be the number requiring surgery each day.

Then $P(Y = r) = \sum_{n=r}^{\infty} P(X = n)P(Y = r|X = n)$

$= \sum_{n=r}^{\infty} \frac{e^{-8}8^n}{n!} \cdot \frac{n!3^{n-r}}{r!(n-r)!4^n}$

$= \frac{e^{-8}}{r!3^r}\sum_{n=r}^{\infty} \frac{6^n}{(n-r)!}$

Writing $k = n - r$,

$P(Y = r) = \frac{e^{-8}}{r!3^r}\sum_{k=0}^{\infty} \frac{6^{k+r}}{k!}$

$= \frac{e^{-8}2^r}{r!}\sum_{k=0}^{\infty} \frac{6^k}{k!}$

$= \frac{e^{-8}2^r}{r!} . e^6$

$= \frac{e^{-2}2^r}{r!}$

so that Y follows a Poisson distribution with mean 2.

(iii) P(8 casualties require surgery on Monday | a total of 12 casualties require surgery on Monday and Tuesday)

= P(8 casualties require surgery on Monday, and a total of 12 casualties require surgery on Monday and Tuesday)

$\div P$(a total of 12 casualties require surgery on Monday and Tuesday)

= P(8 casualties require surgery on Monday, and 4 casualties require surgery on Tuesday)

$\div P$(a total of 12 casualties require surgery on Monday and Tuesday)

= P(8 casualties require surgery on Monday)P(4 casualties require surgery on Tuesday)

$\div P$(a total of 12 casualties require surgery on Monday and Tuesday)

The total number of casualties who require surgery on Monday and Tuesday follows a Poisson distribution with mean $2 \times 2 = 4$

So required prob. $= \frac{\left(\frac{e^{-2}2^8}{8!}\right)\left(\frac{e^{-2}2^4}{4!}\right)}{\left(\frac{e^{-4}4^{12}}{12!}\right)}$

$= \frac{2^{12}12!}{4^{12}8!4!}$

$= \frac{12(11)(10)(9)}{2^{12}(4!)}$

$= \frac{(11)(5)(9)}{2^{12}}$

$= \frac{495}{4096}$