STEP 2014, P3, Q9 - Solution (3 pages; 2/6/20)

1st part

$\underline{v} = \frac{d}{dt}\underline{r} = \frac{k-ke^{-kt}}{k^2}\underline{g} + \frac{ke^{-kt}}{k}\underline{u}$ (as $\underline{g}$ & $\underline{u}$ are constant)

$= \frac{1-e^{-kt}}{k}\underline{g} + e^{-kt}\underline{u}$

To check that the eq'n of motion is satisfied:

N2L $\Rightarrow m\underline{g} - mk\underline{v} = m\underline{a}$, or $\underline{g} - k\underline{v} = \underline{a}$ (1),

where $\underline{a} = \frac{d}{dt}\underline{v} = e^{-kt}\underline{g} - ke^{-kt}\underline{u}$

LHS of (1) is $\underline{g} - k\left(\frac{1-e^{-kt}}{k}\underline{g} + e^{-kt}\underline{u}\right) = \underline{a}$, as required.

To check that the initial conditions are satisfied:

When $t = 0, \underline{v} = \frac{1-1}{k}\underline{g} + \underline{u} = \underline{u}$,

and $\underline{r} = \frac{0-1+1}{k^2}\underline{g} + \frac{1-1}{k}\underline{u} = \underline{0}$, as required.

2nd part

[$\underline{r}\,.\underline{j} = 0 \Rightarrow$ particle crosses the x-axis]

$\underline{r}\,.\underline{j} = 0 \Rightarrow \left(\frac{kT-1+e^{-kT}}{k^2}\right)(-g) + \left(\frac{1-e^{-kT}}{k}\right)(usin\alpha) = 0$

$\Rightarrow (kT - 1 + e^{-kT})(-g) + (1 - e^{-kT})(uksin\alpha) = 0$

$\Rightarrow uksin\alpha = \frac{(kT-1+e^{-kT})g}{1-e^{-kT}} = \left(\frac{kT}{1-e^{-kT}} - 1\right)g,$ as required.

3rd part

At time T, $\underline{v} = \frac{1-e^{-kT}}{k}\underline{g} + e^{-kT}\underline{u}$

$= e^{-kT}ucos\alpha\underline{i} + \left\{(\frac{1-e^{-kT}}{k}\right)(-g) + usin\alpha. e^{-kT}\}\underline{j}$

The particle crosses the x-axis at time T, and as it is moving towards the x-axis, β is the angle below the x-axis, so that

$$tan\beta = \frac{-\left\{\left(\frac{1-e^{-kT}}{k}\right)(-g)+usin\alpha.e^{-kT}\right\}}{e^{-kT}ucos\alpha}$$

$$= \frac{(1-e^{-kT})g}{e^{-kT}ukcos\alpha} - tan\alpha$$

$= \frac{(e^{kT}-1)g}{ukcos\alpha} - tan\alpha$, as required.

4th part

result to prove: $\frac{(e^{kT}-1)g}{ukcos\alpha} - 2tan\alpha > 0$ (1)

From the 2nd part,

$$\text{LHS} = \frac{(e^{kT}-1)}{ukcos\alpha}.\frac{uksin\alpha}{\left(\frac{kT}{1-e^{-kT}}-1\right)} - 2tan\alpha$$

$$= \frac{(e^{kT}-1)(1-e^{-kT})tan\alpha}{kT-(1-e^{-kT})} - 2tan\alpha$$

So (1) is equivalent to $\frac{e^{kT}-1-1+e^{-kT}}{kT-1+e^{-kT}} - 2 > 0$ [as $tan\alpha > 0$] (2)

$$\text{LHS} = \frac{e^{kT}-2+e^{-kT}-2kT+2-2e^{-kT}}{kT-1+e^{-kT}}$$

$= \frac{e^{kT}-e^{-kT}-2kT}{kT-1+e^{-kT}}$ (3)

Numerator of (3) $= 2sinhkT - 2kT > 0$, assuming that

$sinhkT > kT$

Denominator of (3) $= f(kT)$,

where $f(x) = x - 1 + e^{-x}$

$f'(x) = 1 - e^{-x} > 0$ for $x > 0$

Then, as $f(0) = 0$, it follows that $f(x) > 0$ for $x > 0$, and hence the denominator of (3) > 0, as k & $T > 0$

Therefore (3) > 0, and hence $tan\beta > tan\alpha$, so that $\beta > \alpha$ (as $tanx$ is an increasing function for $0 \leq x \leq \frac{\pi}{2}$, and $0 < \alpha < \frac{\pi}{2}$ (given) and $0 \leq \beta \leq \frac{\pi}{2}$ (from the motion of the particle).