STEP 2014, P3, Q7 - Sol'n (4 pages; 16/4/25)

7 The four distinct points P_i $(i = 1, 2, 3, 4)$ are the vertices, labelled anticlockwise, of a cyclic quadrilateral. The lines P_1P_3 and P_2P_4 intersect at Q.

(i) By considering the triangles P_1QP_4 and P_2QP_3 show that $(P_1Q)(QP_3) = (P_2Q)(QP_4)$.

(ii) Let $\mathbf{p}_i$ be the position vector of the point P_i $(i = 1, 2, 3, 4)$. Show that there exist numbers a_i, not all zero, such that

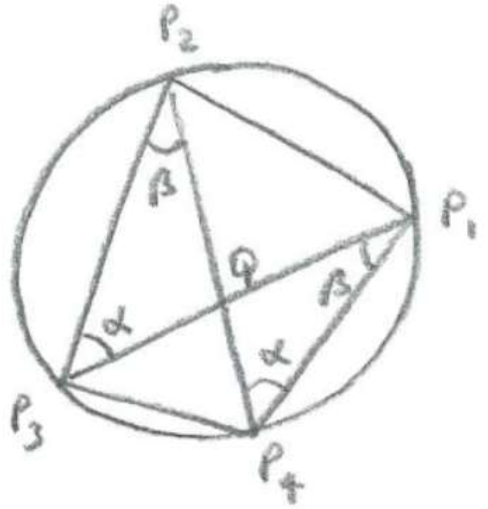
$$\sum_{i=1}^{4} a_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{4} a_i\mathbf{p}_i = \mathbf{0}. \qquad (*)$$

(iii) Let a_i $(i = 1, 2, 3, 4)$ be any numbers, not all zero, that satisfy $(*)$. Show that $a_1 + a_3 \neq 0$ and that the lines P_1P_3 and P_2P_4 intersect at the point with position vector

$$\frac{a_1\mathbf{p}_1 + a_3\mathbf{p}_3}{a_1 + a_3}.$$

Deduce that $a_1a_3(P_1P_3)^2 = a_2a_4(P_2P_4)^2$.

(i)



Referring to the diagram, the chord P_1P_2 subtends the angle α at both P_3 and P_4 (property of a circle). Similarly, the chord P_3P_4 subtends the angle β at both P_1 and P_2. Thus the triangles P_1QP_4 and P_2QP_3 are similar, and hence $\frac{P_1Q}{QP_4} = \frac{P_2Q}{QP_3}$, so that

$(P_1Q)(QP_3) = (P_2Q)(QP_4)$, as required.

(ii) As Q lies on the line segment P_1P_3 , $\underline{q} = \lambda\underline{p}_1 + (1-\lambda)\underline{p}_3$ for some $\lambda > 0$.

Similarly, $\underline{q} = \mu\underline{p}_2 + (1-\mu)\underline{p}_4$

Hence $\lambda\underline{p}_1 + (1-\lambda)\underline{p}_3 = \mu\underline{p}_2 + (1-\mu)\underline{p}_4$,

so that $\lambda\underline{p}_1 - \mu\underline{p}_2 + (1-\lambda)\underline{p}_3 - (1-\mu)\underline{p}_4 = \underline{0}$ (**)

and $\lambda + (-\mu) + (1-\lambda) + [-(1-\mu)] = 0$, as required.

(iii) **1st part**

Suppose that $a_1 + a_3 = 0$,

Then, as $a_1 + a_2 + a_3 + a_4 = 0$, it follows that $a_2 + a_4 = 0$

And so $a_1\underline{p}_1 + a_2\underline{p}_2 + a_3\underline{p}_3 + a_4\underline{p}_4 = \underline{0} \Rightarrow$

$a_1\left(\underline{p}_1 - \underline{p}_3\right) + a_2(\underline{p}_2 - \underline{p}_4) = \underline{0}$,

and hence $a_2(\underline{p}_2 - \underline{p}_4) = -a_1\left(\underline{p}_1 - \underline{p}_3\right)$,

But the P_i are distinct points, and P_2P_4 and P_1P_3 are not parallel,

which means that $a_1 = a_2 = 0$. But this means that all the a_i are

zero, so that we have a contradiction. Hence $a_1 + a_3 \neq 0$.

2nd part

We need to show that the point with position vector $\frac{a_1\underline{p}_1 + a_3\underline{p}_3}{a_1 + a_3}$

lies on both P_1P_3 and P_2P_4.

Clearly it lies on P_1P_3, as any point on P_1P_3 can be written as

$\lambda\underline{p}_1 + (1 - \lambda)\underline{p}_3$ for a suitable λ. (So here $\lambda = \frac{a_1}{a_1 + a_3}$)

Also, $\frac{a_1\underline{p}_1 + a_3\underline{p}_3}{a_1 + a_3} = \frac{-(a_2\underline{p}_2 + a_4\underline{p}_4)}{-(a_2 + a_4)} = \frac{a_2\underline{p}_2 + a_4\underline{p}_4}{a_2 + a_4}$, so that it lies on P_2P_4 as well (at $\mu\underline{p}_2 + (1 - \mu)\underline{p}_4$, with $\mu = \frac{a_2}{a_2 + a_4}$), as required.

3rd part

[Note that $(P_1P_3)^2 = \left(\underline{p}_1 - \underline{p}_3\right).\left(\underline{p}_1 - \underline{p}_3\right)$, and that the result from (i) is almost certainly to be used. The question is, whether to start from the result from (i), and try to obtain

$a_1a_3(P_1P_3)^2 = a_2a_4(P_2P_4)^2$, or the other way round. Trying the 1st approach:]

$(P_1Q)(QP_3) = (P_2Q)(QP_4) \Rightarrow (P_1Q)^2(QP_3)^2 = (P_2Q)^2(QP_4)^2$

LHS $= \left(\underline{p}_1 - \underline{q}\right).\left(\underline{p}_1 - \underline{q}\right)\left(\underline{p}_3 - \underline{q}\right).\left(\underline{p}_3 - \underline{q}\right)$ (***)

Now from the 2nd part of (iii),

$\underline{p}_1 - \underline{q} = \underline{p}_1 - \frac{a_1\underline{p}_1 + a_3\underline{p}_3}{a_1 + a_3}$

Writing $A = a_1 + a_3$, this equals $\frac{\underline{p}_1(A - a_1) - a_3\underline{p}_3}{A} = \frac{a_3}{A}(\underline{p}_1 - \underline{p}_3)$

Similarly, $\underline{p}_3 - \underline{q} = \frac{a_1}{A}(\underline{p}_3 - \underline{p}_1)$,

and hence (***) equals

$\frac{a_3}{A}\left(\underline{p}_1 - \underline{p}_3\right).\frac{a_3}{A}\left(\underline{p}_1 - \underline{p}_3\right)\frac{a_1}{A}\left(\underline{p}_3 - \underline{p}_1\right).\frac{a_1}{A}(\underline{p}_3 - \underline{p}_1)$

$= \frac{{a_3}^2{a_1}^2}{A^4}\left|\underline{p}_1 - \underline{p}_3\right|^4$

Similarly, RHS equals $\frac{{a_4}^2{a_2}^2}{B^4}\left|\underline{p}_2 - \underline{p}_4\right|^4$,

where $B = a_2 + a_4 = -(a_1 + a_3) = -A$,

and so, taking the square root of each side,

$a_1a_3(P_1P_3)^2 = a_2a_4(P_2P_4)^2$, as required.

[Trying the other way:

$a_1a_3(P_1P_3)^2 = a_1a_3\left(\underline{p}_1 - \underline{p}_3\right).\left(\underline{p}_1 - \underline{p}_3\right)$,

but it isn't obvious how to introduce $\underline{q}$]