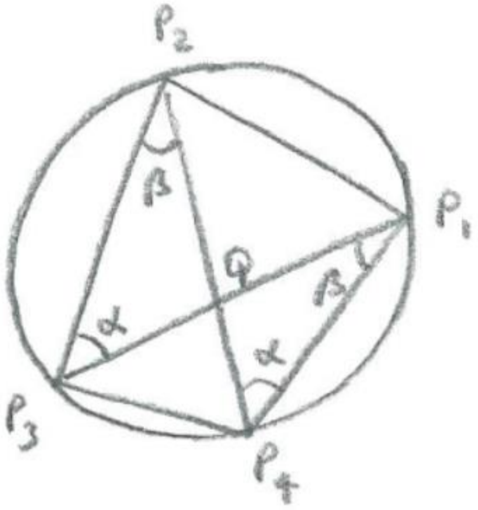
STEP 2014, P3, Q7 - Sol'n (3 pages; 11/3/21)

(i)



Referring to the diagram, the chord P_1P_2 subtends the angle α at both P_3 and P_4 (property of a circle). Similarly, the chord P_3P_4 subtends the angle β at both P_1 and P_2. Thus the triangles P_1QP_4 and P_2QP_3 are similar, and hence $\frac{P_1Q}{QP_4} = \frac{P_2Q}{QP_3}$, so that

$(P_1Q)(QP_3) = (P_2Q)(QP_4)$, as required.

(ii) As Q lies on the line segment P_1P_3 , $\underline{q} = \lambda\underline{p}_1 + (1-\lambda)\underline{p}_3$ for some $\lambda > 0$.

Similarly, $\underline{q} = \mu\underline{p}_2 + (1-\mu)\underline{p}_4$

Hence $\lambda\underline{p}_1 + (1-\lambda)\underline{p}_3 = \mu\underline{p}_2 + (1-\mu)\underline{p}_4$,

so that $\lambda\underline{p}_1 - \mu\underline{p}_2 + (1-\lambda)\underline{p}_3 - (1-\mu)\underline{p}_4 = \underline{0}$ (**)

and $\lambda + (-\mu) + (1-\lambda) + [-(1-\mu)] = 0$, as required.

(iii) **1st part**

[Assuming that the $\underline{p}_i$ are the position vectors of the P_i defined at the start; ie they are not just any old vectors that satisfy (*).]

As the $\underline{p}_i$ are not parallel, the ratios of the a_i are uniquely determined by (**).

So $a_1 = k\lambda$ and $a_3 = k(1-\lambda)$, for some $k \neq 0$

Then $a_1 + a_3 = k[\lambda + (1-\lambda)] = k \neq 0$

2nd part

The line (segments) P_1P_3 and P_2P_4 intersect at Q.

And $\frac{a_1\underline{p}_1 + a_3\underline{p}_3}{a_1+a_3} = \frac{k\lambda\underline{p}_1 + k(1-\lambda)\underline{p}_3}{k\lambda + k(1-\lambda)} = \lambda\underline{p}_1 + (1-\lambda)\underline{p}_3 = \underline{q}$,

which is the position vector of Q, as required.

3rd part

[Note that $(P_1P_3)^2 = \left(\underline{p}_1 - \underline{p}_3\right).\left(\underline{p}_1 - \underline{p}_3\right)$, and that the result from (i) is almost certainly to be used. The question is, whether to start from the result from (i), and try to obtain

$a_1a_3(P_1P_3)^2 = a_2a_4(P_2P_4)^2$, or the other way round. Trying the 1st approach:]

$(P_1Q)(QP_3) = (P_2Q)(QP_4) \Rightarrow (P_1Q)^2(QP_3)^2 = (P_2Q)^2(QP_4)^2$

LHS = $\left(\underline{p}_1 - \underline{q}\right).\left(\underline{p}_1 - \underline{q}\right)\left(\underline{p}_3 - \underline{q}\right).\left(\underline{p}_3 - \underline{q}\right)$ (***)

Now from the 2nd part of (iii),

$\underline{p}_1 - \underline{q} = \underline{p}_1 - \frac{a_1\underline{p}_1 + a_3\underline{p}_3}{a_1+a_3}$

Writing $A = a_1 + a_3$, this equals $\frac{\underline{p}_1(A-a_1) - a_3\underline{p}_3}{A} = \frac{a_3}{A}(\underline{p}_1 - \underline{p}_3)$

Similarly, $\underline{p}_3 - \underline{q} = \frac{a_1}{A}(\underline{p}_3 - \underline{p}_1)$,

and hence (***) equals

$\frac{a_3}{A}\left(\underline{p}_1 - \underline{p}_3\right).\frac{a_3}{A}\left(\underline{p}_1 - \underline{p}_3\right)\frac{a_1}{A}\left(\underline{p}_3 - \underline{p}_1\right).\frac{a_1}{A}(\underline{p}_3 - \underline{p}_1)$

$= \frac{{a_3}^2{a_1}^2}{A^4}\left|\underline{p}_1 - \underline{p}_3\right|^4$

Similarly, RHS equals $\frac{{a_4}^2{a_2}^2}{B^4}\left|\underline{p}_2 - \underline{p}_4\right|^4$,

where $B = a_2 + a_4 = -(a_1 + a_3) = -A$,

and so, taking the square root of each side,

$a_1 a_3 (P_1 P_3)^2 = a_2 a_4 (P_2 P_4)^2$, as required.

[Trying the other way:

$a_1 a_3 (P_1 P_3)^2 = a_1 a_3 \left(\underline{p}_1 - \underline{p}_3\right).\left(\underline{p}_1 - \underline{p}_3\right)$,

but it isn't obvious how to introduce $\underline{q}$]