STEP 2014, P3, Q2 - Solution (2 pages; 28/5/20)

(i) $cosh\,2x = cosh^2x + sinh^2x = cosh^2x + (cosh^2x - 1)$

If $u = coshx, du = sinhx\;dx$,

and so $\int \frac{sinhx}{\cosh 2x}\;dx = \int \frac{1}{2u^2-1}du$

$= \frac{1}{2}\int \frac{\frac{1}{\sqrt{2}}}{(u-\frac{1}{\sqrt{2}})} - \frac{\frac{1}{\sqrt{2}}}{(u+\frac{1}{\sqrt{2}})}\;dx$

$= \frac{1}{2\sqrt{2}} ln\left|\frac{coshx-\frac{1}{\sqrt{2}}}{coshx+\frac{1}{\sqrt{2}}}\right| + C = \frac{1}{2\sqrt{2}} ln\left|\frac{\sqrt{2}coshx-1}{\sqrt{2}coshx+1}\right| + C$, as required.

(ii) $cosh\,2x = cosh^2x + sinh^2x = (sinh^2x + 1) + sinh^2x$

If $u = sinhx, du = coshx\;dx$,

and so $\int \frac{coshx}{\cosh 2x}\;dx = \int \frac{1}{2u^2+1}du$

$= \frac{1}{2}.\frac{1}{\left(\frac{1}{\sqrt{2}}\right)}\text{arctan}\left(\frac{u}{\left(\frac{1}{\sqrt{2}}\right)}\right) + C$

$= \frac{\sqrt{2}}{2}\text{arctan}\left(\sqrt{2}sinhx\right) + C$

(iii) $I = \int \frac{sinhx}{\cosh 2x}\;dx = \int \frac{\frac{1}{2}(e^x-e^{-x})}{\frac{1}{2}(e^{2x}+e^{-2x})}dx$

and $J = \int \frac{coshx}{\cosh 2x}\;dx = \int \frac{\frac{1}{2}(e^x+e^{-x})}{\frac{1}{2}(e^{2x}+e^{-2x})}dx$

$K = \int \frac{1}{1+u^4}du = \int \frac{u^{-2}}{u^{-2}+u^2}du$

Let $u = e^x$, so that $du = e^x dx$,

and $K = \int \frac{e^{-2x}.e^{x}}{e^{-2x}+e^{2x}}dx = \int \frac{e^{-x}}{e^{-2x}+e^{2x}}dx = \frac{1}{2}(J - I)$

So $\int_0^1 \frac{1}{1+u^4}\ du = \frac{1}{2}[\frac{\sqrt{2}}{2}\arctan\left(\sqrt{2}sinhx\right) - \frac{1}{2\sqrt{2}}ln\left|\frac{\sqrt{2}coshx-1}{\sqrt{2}coshx+1}\right|]_{-\infty}^{0}$

$= \left(0 - \frac{1}{4\sqrt{2}}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right) - \left(\frac{\sqrt{2}}{4}\left(-\frac{\pi}{2}\right) - \frac{1}{4\sqrt{2}}\ln(1)\right)$

$= \frac{1}{4\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) + \frac{\pi}{4\sqrt{2}}$

$= \frac{1}{4\sqrt{2}}\ln(\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{2-1}) + \frac{\pi}{4\sqrt{2}}$

$= \frac{\pi}{4\sqrt{2}} + \frac{2}{4\sqrt{2}}\ln(\sqrt{2}+1)$

$= \frac{\pi+2\ln(\sqrt{2}+1)}{4\sqrt{2}}$, as required.