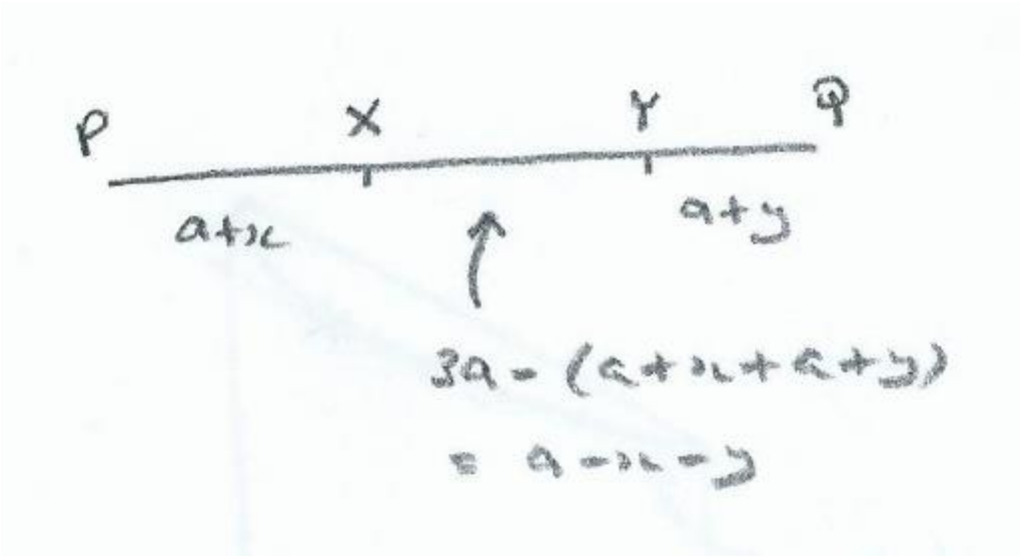
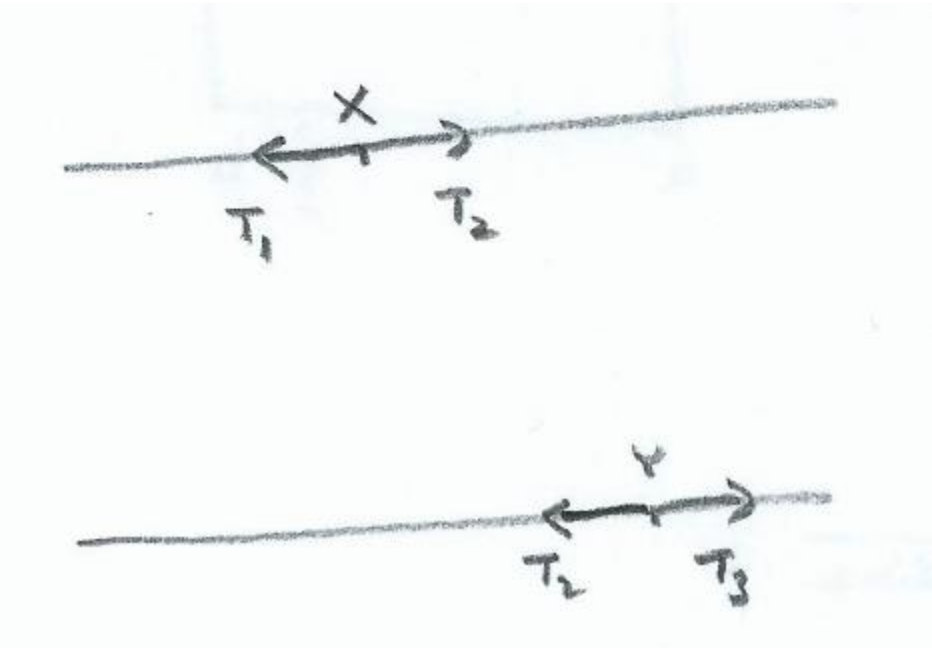
STEP 2014, P3, Q10 - Solution (3 pages; 29/5/20)

1st part





Referring to the diagrams, $T_1 = \frac{\lambda x}{a}$, $T_2 = -\frac{\lambda(x+y)}{a}$ & $T_3 = \frac{\lambda y}{a}$, by Hooke's law (thus $T_2 < 0$ and the force is a compression).

Then, applying N2L to X (noting that x increases as X moves to the right)

$m\frac{d^2x}{dt^2} = T_2 - T_1 = \frac{\lambda}{a}(-x - y - x) = -\frac{\lambda}{a}(2x + y)$, as required.

2nd part

For Y (noting that y increases as Y moves to the left):

$m\frac{d^2y}{dt^2} = T_2 - T_3 = \frac{\lambda}{a}(-x - y - y) = -\frac{\lambda}{a}(2y + x)$

[as would be expected from the symmetry]

3rd part

Hence $m\frac{d^2(x-y)}{dt^2} = -\frac{\lambda}{a}(2x + y - 2y - x) = -\frac{\lambda}{a}(x - y)$

$\Rightarrow$ SHM of $x - y$, with $x - y = Ccos(\omega t - \theta)$,

where $\omega^2 = \frac{\lambda}{ma}$

[The official sol'ns don't suggest that the SHM differential eq'n needs to be solved from 1st principles. Usually standard results such as this can just be quoted.]

When $t = 0, x = 0$ & $y = -\frac{a}{2}$, so that $\frac{a}{2} = Ccos\theta$

Also, $\frac{d(x-y)}{dt} = -\omega Csin(\omega t - \theta)$,

and initially $\frac{dx}{dt} = \frac{dy}{dt} = 0$, so that $0 = \omega Csin\theta$, and hence $\theta = 0$,

and so $C = \frac{a}{2}$, giving $x - y = \frac{a}{2}cos(\omega t)$

4th part

Adding the differential eq'ns instead,

$m\frac{d^2(x+y)}{dt^2} = -\frac{\lambda}{a}(2x + y + 2y + x) = -\frac{3\lambda}{a}(x + y)$,

so that $x + y = Dcos(\omega_1 t - \phi)$, where ${\omega_1}^2 = \frac{3\lambda}{ma}$

When $t = 0, x = 0$ & $y = -\frac{a}{2}$, so that $-\frac{a}{2} = Dcos\phi$

Also, $\frac{d(x+y)}{dt} = -\omega Dsin(\omega_1 t - \phi)$,

and initially $\frac{dx}{dt} = \frac{dy}{dt} = 0$, so that $0 = \omega Dsin\phi$, and hence $\phi = 0$,

and so $D = -\frac{a}{2}$, giving $x + y = -\frac{a}{2}cos(\omega_1 t)$

5th part

Subtracting $x - y = \frac{a}{2}cos(\omega t)$ from $x + y = -\frac{a}{2}cos(\omega_1 t)$:

$2y = -\frac{a}{2}(cos(\omega_1 t) + cos(\omega t))$

Then $y = -\frac{a}{2} \Rightarrow cos(\omega_1 t) + cos(\omega t) = 2$

$\Rightarrow cos(\omega_1 t) = cos(\omega t) = 1$

$\Rightarrow t = 0$ or $\omega_1 t = 2m\pi$ and $\omega t = 2n\pi$ (where m & n are integers)

Then $\frac{\omega_1 t}{\omega t} = \frac{2m\pi}{2n\pi}$, and hence $\frac{\omega_1}{\omega} = \frac{m}{n}$ (a rational number).

But $\omega^2 = \frac{\lambda}{ma}$ & $\omega_1{}^2 = \frac{3\lambda}{ma}$, so that $\frac{\omega_1}{\omega} = \sqrt{3}$ (an irrational number).

So $y = -\frac{a}{2}$ only when $t = 0$; ie it never returns to its initial position.