STEP 2014, P2, Q12 - Solution (4 pages; 25/6/20)

(i) P(dying within following δt | fly lives to at least time t)

$= \frac{P(\text{fly lives to at least time t and dies within following } \delta t)}{P(\text{fly lives to at least time } t)}$

$= \frac{F(t+\delta t)-F(t)}{1-F(t)}$ (A)

Now $f(t) = \lim_{\delta t \to 0} \frac{F(t+\delta t)-F(t)}{\delta t}$,

so that $F(t+\delta t) - F(t) \approx \text{f(t)}\delta t$ for small δt.

Also $h(t) = \frac{f(t)}{1-F(t)}$,

so that (A) $\approx \frac{\text{f(t)}\delta t}{\left(\frac{f(t)}{h(t)}\right)} = h(t)\delta t$, as required.

(ii) When $F(t) = \frac{t}{a}$, $f(t) = \frac{d}{dt}\left(\frac{t}{a}\right) = \frac{1}{a}$

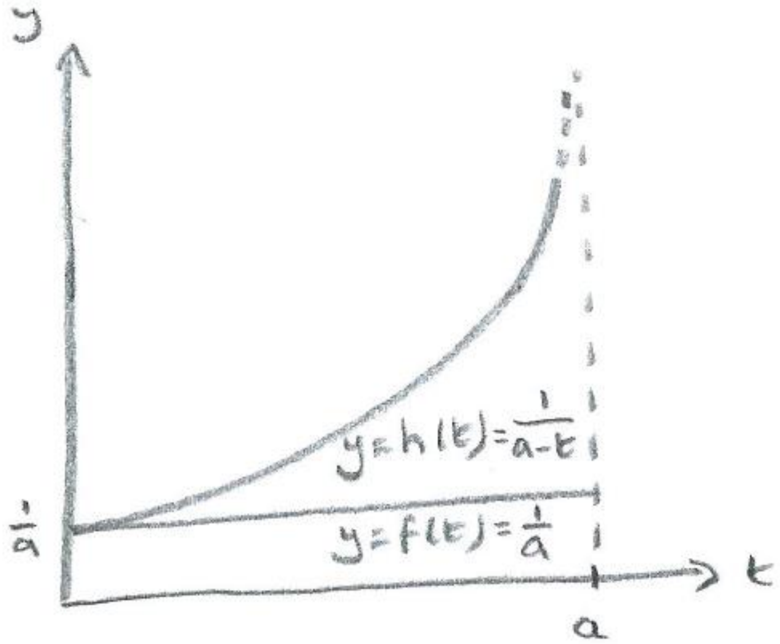
so that $h(t) = \frac{f(t)}{1-F(t)} = \frac{\left(\frac{1}{a}\right)}{1-\left(\frac{t}{a}\right)} = \frac{1}{a-t}$

$h(t) = \frac{1}{a-t}$ can be obtained from $f(t) = \frac{1}{t}$ by the following transformations:

translation of $\begin{pmatrix} -a \\ 0 \end{pmatrix}$ (to give $\frac{1}{t+a}$),

followed by relection in y-axis (to give $\frac{1}{-t+a}$)

Note that $h'(t) = \frac{1}{(a-t)^2}$, so that $h'(0) = \frac{1}{a^2}$



(iii) $\frac{1}{t} = \frac{f(t)}{1-F(t)} \Rightarrow \int_a^T \frac{1}{t}\,dt = \int_a^T \frac{f(t)}{1-F(t)}\,dt$

$\Rightarrow \ln\left(\frac{T}{a}\right) = [-\ln(1-F(t))]_a^T$, as $f(t) = \frac{d}{dt}F(t)$

$= -\ln\left(1-F(T)\right) + \ln(1-F(a))$

$= \ln\left(\frac{1}{1-F(T)}\right)$, as $F(a) = 0$

Hence $\frac{T}{a} = \frac{1}{1-F(T)}$, so that $1 - F(T) = \frac{a}{T}$ and $F(t) = 1 - \frac{a}{t}$

Then $f(t) = \frac{d}{dt}\left(1 - \frac{a}{t}\right) = \frac{a}{t^2}$

[Check: $h(t) = \frac{f(t)}{1-F(t)} = \frac{\left(\frac{a}{t^2}\right)}{\left(\frac{a}{t}\right)} = \frac{1}{t}$]

(iv) Suppose that $h(t) = c$ (a constant) for $t > b$, and zero otherwise.

Then, for $T > b$, $\int_b^T c\, dt = \int_b^T \frac{f(t)}{1-F(t)}\, dt = [-\ln(1 - F(t))]_b^T$,

so that $c(T - b) = -\ln\left(1 - F(T)\right) + \ln(1 - F(b))$

$= ln\left(\frac{1-F(b)}{1-F(T)}\right)$

and $e^{c(t-b)} = \frac{1-F(b)}{1-F(t)}$,

so that $1 - F(t) = (1 - F(b))e^{-c(t-b)}$

and differentiating:

$-f(t) = (1 - F(b))e^{-c(t-b)}(-c)$

so that $f(t) = c(1 - F(b))e^{-c(t-b)}$ for $t > b$ (A)

Also, as $h(t) = \frac{f(t)}{1-F(t)}$ for $F(t) < 1$, and $h(t) = 0$ for $t \leq b$,

it follows that $f(t) = 0$ for $t \leq b$, and hence $F(b) = 0$.

So, from (A), $f(t) = ce^{-c(t-b)}$ for $t > b$,

and replacing c with k gives the required form,

and k has to be positive, in order for $f(t)$ to be positive.

Suppose instead that $f(t) = ke^{-k(t-b)}$ for $t > b$ (where k is a positive constant).

Then $F(t) = C - e^{-k(t-b)}$

As $t \to \infty, F(t) \to 1$, so that $C = 1$.

Then $h(t) = \frac{f(t)}{1-F(t)} = \frac{ke^{-k(t-b)}}{e^{-k(t-b)}} = k$, for $t > b$, as required.

(v) $\left(\frac{\lambda}{\theta^\lambda}\right) t^{\lambda-1} = \frac{f(t)}{1-F(t)} \Rightarrow \frac{\lambda}{\theta^\lambda}\int_0^T t^{\lambda-1}\, dt = \int_0^T \frac{f(t)}{1-F(t)}\, dt$

$\Rightarrow \frac{\lambda}{\theta^\lambda}\left[\frac{1}{\lambda}t^\lambda\right]\begin{matrix} T \\ 0 \end{matrix} = [-\ln(1 - F(t))]_0^T$

$\Rightarrow \frac{1}{\theta^\lambda}\left(T^\lambda\right) = -ln\big(1 - F(T)\big)$, as $F(0) = 0$

Hence $1 - F(T) = e^{-\left(\frac{T}{\theta}\right)^\lambda}$

and $F(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^\lambda}$

Then $f(t) = \lambda\left(\frac{t}{\theta}\right)^{\lambda-1} e^{-\left(\frac{t}{\theta}\right)^\lambda}$