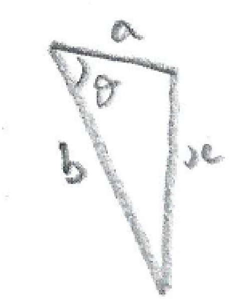
STEP 2014, P1, Q4 - Solution (2 pages; 9/11/19)



$x^2 = a^2 + b^2 - 2abcos\theta$ (1)

Differentiating wrt t: $2x\frac{dx}{dt} = 2absin\theta\frac{d\theta}{dt}$ (2)

Differentiating again wrt t:

$2(\frac{dx}{dt})^2 + 2x\frac{d^2x}{dt^2} = 2abcos\theta(\frac{d\theta}{dt})^2 + 2absin\theta\frac{d^2\theta}{dt^2}$ (3)

The rate of increase of x is greatest when $\frac{d^2x}{dt^2} = 0$

Then (3) $\Rightarrow (\frac{dx}{dt})^2 = abcos\theta(\frac{d\theta}{dt})^2 + absin\theta\frac{d^2\theta}{dt^2}$

whilst (2) $\Rightarrow (\frac{dx}{dt})^2 = (\frac{absin\theta}{x}\frac{d\theta}{dt})^2$

Hence $abcos\theta(\frac{d\theta}{dt})^2 + absin\theta\frac{d^2\theta}{dt^2} = (\frac{absin\theta}{x}\ \frac{d\theta}{dt})^2$ (4)

At time t minutes from midday,

$\theta = \frac{t}{60}(2\pi) - \frac{t}{60}\left(\frac{2\pi}{12}\right) = \frac{11t(2\pi)}{12(60)} = \frac{11\pi t}{360}$ (5)

so that $\frac{d\theta}{dt} = \frac{11\pi}{360}$ and $\frac{d^2\theta}{dt^2} = 0$

Then (4) $\Rightarrow abcos\theta = (\frac{absin\theta}{x})^2$

$\Rightarrow x^2 = \frac{ab(1-cos^2\theta)}{cos\theta}$

Then (1) $\Rightarrow a^2 + b^2 - 2abcos\theta = \frac{ab(1-cos^2\theta)}{cos\theta}$

$\Rightarrow (a^2 + b^2)C - 2abC^2 = ab(1 - C^2)$, where $C = cos\theta$

$\Rightarrow abC^2 - (a^2 + b^2)C + ab = 0$

$\Rightarrow (aC - b)(bC - a) = 0$

[$A + B = -(a^2 + b^2)$ & $AB = (ab)(ab)$

$\Rightarrow A = -a^2, B = -b^2, say$]

$\Rightarrow C = \frac{b}{a}$ or $\frac{a}{b}$; reject $\frac{b}{a}$, as $b > a$

[or from the quadratic formula]

Then (1) $\Rightarrow x^2 = a^2 + b^2 - 2ab\left(\frac{a}{b}\right) = b^2 - a^2$

$\Rightarrow x = (b^2 - a^2)^{\frac{1}{2}}$, as required.

When $b = 2a$, the greatest rate of increase occurs when

$cos\theta = \frac{a}{b} = \frac{1}{2}$, so that $\theta = \frac{\pi}{3}$ (as $0 \leq \theta < \pi$)

Then, as $\theta = \frac{11\pi t}{360}$ (from (5)), $t = \frac{\pi}{3} . \frac{360}{11\pi} = \frac{120}{11}$,

which is just less than $\frac{121}{11} = 11$ minutes, as required.