STEP 2014, P1, Q3 - Solution (3 pages; 8/11/19)

(i) $\left[\frac{1}{3}x^3\right]_0^b = ([\frac{1}{2}x^2]_0^b)^2 \Rightarrow \frac{b^3}{3} = (\frac{b^2}{2})^2$

$\Rightarrow 4b^3 - 3b^4 = 0 \Rightarrow 3b - 4 = 0$ (as $b \neq 0$) $\Rightarrow b = \frac{4}{3}$

(ii) $a = 1 \Rightarrow \left[\frac{1}{3}x^3\right]_1^b = ([\frac{1}{2}x^2]_1^b)^2 \Rightarrow \frac{1}{3}(b^3 - 1) = \frac{1}{4}(b^2 - 1)^2$

$\Rightarrow 4b^3 - 4 = 3(b^4 - 2b^2 + 1)$

$\Rightarrow 3b^4 - 4b^3 - 6b^2 + 7 = 0$; say $f(b) = 0$

We can look for a factorisation, and from the first and last terms of the cubic it would have to be of the form

$3b^4 - 4b^3 - 6b^2 + 7 = (b - 1)(3b^3 + Ab^2 + Bb - 7)$

To verify this, applying the Factor theorem,

$f(1) = 3 - 4 - 6 + 7 = 0$

[Also, as the official sol'n points out, the two integrals would both be zero when $a = b$, so that $a = b = 1$ must be a solution.]

Then equating coefficients of b^3 gives $-4 = A - 3$, so that

$A = -1$; and equating coefficients of b^2 gives $-6 = B - A$,

so that $B = -7$.

Thus $f(b) = (b - 1)(3b^3 - b^2 - 7b - 7)$,

and as $b > a = 1$, $f(b) = 0 \Rightarrow 3b^3 - b^2 - 7b - 7 = 0$, as required.

To establish that there is only one real root of the cubic:

Write $g(b) = 3b^3 - b^2 - 7b - 7$

Consider the turning points of the graph:

$g'(b) = 0 \Rightarrow 9b^2 - 2b - 7 = 0$

$\Rightarrow (9b + 7)(b - 1) = 0 \Rightarrow b = -\frac{7}{9}\ or\ 1$

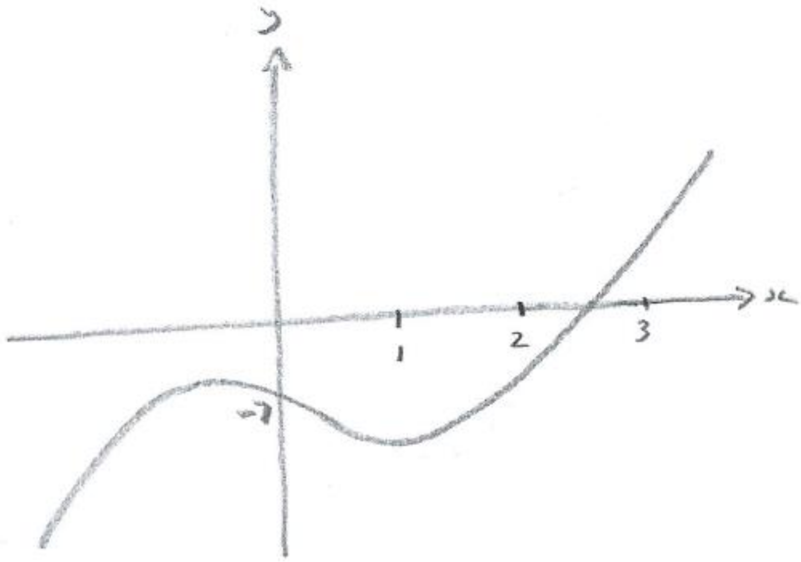
Then $g(1) = 3 - 1 - 7 - 7 < 0$

There will only be one real root if both turning points lie below the x-axis.

$g\left(-\frac{7}{9}\right) = -3\left(\frac{7}{9}\right)^3 - \left(\frac{7}{9}\right)^2 + 7\left(\frac{7}{9}\right) - 7$

$= -3\left(\frac{7}{9}\right)^3 - \left(\frac{7}{9}\right)^2 - 7\left(\frac{2}{9}\right) < 0$

Thus there is only one real root.



[To aid with sketching, cubics have rotational symmetry about their point of inflexion, which is halfway between the turning points. See Pure/Graphs/"Cubic Functions"]

Then $g(2) = 24 - 4 - 14 - 7 = -1 < 0$

and $g(3) = 81 - 9 - 21 - 7 = 44 > 0$,

so that the root lies between 2 and 3.

(iii)) $\left[\frac{1}{3}x^3\right]_a^b = (\left[\frac{1}{2}x^2\right]_a^b)^2 \Rightarrow \frac{1}{3}(b^3 - a^3) = \frac{1}{4}(b^2 - a^2)^2$

$\Rightarrow 4(b-a)(b^2 + ab + a^2) = 3(b-a)^2(b+a)^2 \quad (1)$

Now $p^2 = b^2 + a^2 + 2ab$ and $q^2 = b^2 + a^2 - 2ab$,

so that $p^2 - q^2 = 4ab$ and $p^2 + q^2 = 2(b^2 + a^2)$

Then (1) $\Rightarrow 2(p^2 + q^2) + (p^2 - q^2) = 3qp^2$ (as $b - a \neq 0$)

$\Rightarrow 3p^2 + q^2 = 3p^2q$, as required.

Then $3p^2(1-q) = -q^2$, so that $p^2 = \frac{q^2}{3(q-1)}$, provided $q \neq 1$

If $q = 1$, then $3p^2 + q^2 = 3p^2q \Rightarrow 3p^2 + 1 = 3p^2$, which is impossible.

As $b > a \geq 0, p = b + a \neq 0$

So $p^2 > 0$, and then $p^2 = \frac{q^2}{3(q-1)} \Rightarrow q > 1$.

From (1), dividing by $b - a = q > 0$,

$\frac{4}{3} = \frac{q(b^2+a^2+2ab)}{b^2+a^2+ab} \geq q$ (as $a \geq 0, b > 0$)

Thus, $1 < b - a \leq \frac{4}{3}$, as required.