STEP 2013, P3, Q9 - Solution (2 pages; 26/6/20)

1st part

Referring to the diagrams, define coordinate axes as follows:

O is the centre of the sphere; X is the distance below O of the infinitesimal disc shown, and Y is its radius.

The required volume V is the volume of the hemisphere less V_1.

$V_1 = \int_0^x \pi Y^2 dX = \pi \int_0^x (R^2 - X^2) dX$

$= \pi \left[R^2 X - \frac{1}{3}X^3\right]_0^x = \pi(R^2 x - \frac{1}{3}x^3)$

So $V = \frac{2}{3}\pi R^3 - \pi(R^2 x - \frac{1}{3}x^3)$

$= \frac{\pi}{3}(2R^3 - 3R^2 x + x^3)$, as required.

2nd part

Noting that x is the distance above the level of the liquid, so that we are treating upwards as the positive direction:

N2L $\Rightarrow V\rho g - \frac{4}{3}\pi R^3 \rho_s g = \frac{4}{3}\pi R^3 \rho_s \ddot{x}$

$\Rightarrow 4R^3 \rho_s (g + \ddot{x}) = \frac{3}{\pi} V \rho g$

$= (2R^3 - 3R^2 x + x^3)\rho g,$ as required. (A)

3rd part

$x = \frac{R}{2}$, $\ddot{x} = 0 \Rightarrow 4R^3\rho_s g = R^3\rho g(2 - \frac{3}{2} + \frac{1}{8})$

$\Rightarrow 32\rho_s = \rho(16 - 12 + 1) = 5\rho$

$\Rightarrow \rho_s = \frac{5\rho}{32}$

4th part

Let $y = x - \frac{R}{2}$ (where y is small).

Then (A) $\Rightarrow 4R^3\left(\frac{5\rho}{32}\right)(g + \ddot{y}) = (2R^3 - 3R^2(y + \frac{R}{2}) + (y + \frac{R}{2})^3)\rho g$

$\Rightarrow \frac{5R^3}{8g}(g + \ddot{y}) = 2R^3 - 3R^2y - \frac{3R^3}{2} + o(y) + \frac{3yR^2}{4} + \frac{R^3}{8}$

[$o(y)$ represents terms of order smaller than y; ie involving y^2 or higher powers of y]

$\Rightarrow 5R^3(g + \ddot{y})$

$= 16R^3g - 24R^2gy - 12R^3g + o(y) + 6yR^2g + R^3g$

$\Rightarrow 5R^3\ddot{y} = -18R^2gy + o(y)$

$\Rightarrow \ddot{y} \approx -\frac{18g}{5R}y$

$\Rightarrow \omega^2 = \frac{18g}{5R}$, where the period of oscillations T satisfies $\omega T = 2\pi$,

so that $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{18g}{5R}}} = 2\pi\sqrt{\frac{5R}{18g}} = \frac{\pi}{3}\sqrt{\frac{10R}{g}}$