STEP 2013, P3, Q7 - Solution (3 pages; 1/7/20)

(i) **1st part**

$\frac{d}{dx}E(x) = 2\frac{dy}{dx}\,.\frac{d^2y}{dx^2} + 2y^3\frac{dy}{dx} = 2\frac{dy}{dx}\left(\frac{d^2y}{dx^2} + y^3\right) = 0$, so that $E(x)$ is constant, as required.

2nd part

$E(0) = 0^2 + \frac{1}{2}\,.1^4 = \frac{1}{2}$, so that $E(x) = \frac{1}{2}$ for all x (as $E(x)$ is constant)

Thus $\left(\frac{dy}{dx}\right)^2 + \frac{1}{2}y^4 = \frac{1}{2}$,

and so $y^4 = 1 - 2\left(\frac{dy}{dx}\right)^2 \leq 1$,

and hence $|y| \leq 1$, as required.

(ii) **1st part**

$\frac{d}{dx}E(x) = 2\frac{dv}{dx}\left(\frac{d^2v}{dx^2}\right) + 2sinhx.\frac{dv}{dx}$

$= 2\frac{dv}{dx}(\frac{d^2v}{dx^2} + sinhx)$

$= 2\frac{dv}{dx}(-x\frac{dv}{dx})$

$= -2x\left(\frac{dv}{dx}\right)^2 \leq 0$ for $x \geq 0$, as required.

2nd part

$2coshv = E(x) - \left(\frac{dv}{dx}\right)^2 \leq E(x) \leq E(0)$, for $x \geq 0$, as $\frac{d}{dx}E(x) \leq 0$

And $E(0) = 0^2 + e^{ln3} + e^{-ln3} = 3 + \frac{1}{3} = \frac{10}{3}$

So $2coshv \leq \frac{10}{3}$, and hence $coshv \leq \frac{5}{3}$ (for $x \geq 0$), as required.

(iii) Trying the approach in the 1st two parts,

let $E(x) = \left(\frac{dw}{dx}\right)^2 + 2\int wcoshw + 2sinhw\, dw \quad (1)$

(where the constant of integration is zero)

Then $\frac{d}{dx}E(x) = 2\left(\frac{dw}{dx}\right).\frac{d^2w}{dx^2} + 2(wcoshw + 2sinhw).\frac{dw}{dx}$

$= 2\frac{dw}{dx}(\frac{d^2w}{dx^2} + wcoshw + 2sinhw)$

$= -2\frac{dw}{dx}(5coshx - 4sinhx - 3)\left(\frac{dw}{dx}\right)$

Result to prove: $f(x) = 5coshx - 4sinhx - 3 \geq 0$, for $x \geq 0$

$f'(x) = 5sinhx - 4coshx$

and so $f'(x) = 0$ when $5sinhx = 4coshx$

$\Rightarrow 25sinh^2x = 16cosh^2x = 16(sinh^2x + 1)$

$\Rightarrow sinh^2x = \frac{16}{9} \Rightarrow sinhx = \frac{4}{3}$ (for $x \geq 0$)

And $f''(x) = 5coshx - 4sinhx$

$= 4(coshx - sinhx) + coshx > coshx > 0$

(as $coshx > sinhx$ for all x)

Hence $f(x)$ has a single (local) minimum when $sinhx = \frac{4}{3}$

Then $cosh^2x = sinh^2x + 1 = \frac{16}{9} + 1 = \frac{25}{9}$, so that $coshx = \frac{5}{3}$,

and the minimum value of $f(x)$ is $5\left(\frac{5}{3}\right) - 4\left(\frac{4}{3}\right) - 3 = \frac{25-16-9}{3} = 0$

So, as there is a single minimum, $f(x) \geq 0$, for $x \geq 0$, as required to be proved.

[Alternatively, $f(x) = \frac{5}{2}(e^x + e^{-x}) - \frac{4}{2}(e^x - e^{-x}) - 3$

$= \frac{1}{2}e^{-x}(e^{2x} + 9 - 6e^x) = \frac{1}{2}e^{-x}(e^x - 3)^2 \geq 0]$

Hence $\frac{d}{dx}E(x) \leq 0$ (for $x \geq 0$)

And from (1),

$E(x) = \left(\frac{dw}{dx}\right)^2 + 2\int wcoshw + 2sinhw\ dw$

$= \left(\frac{dw}{dx}\right)^2 + 2wsinhw - 2coshw + 4coshw$

$= \left(\frac{dw}{dx}\right)^2 + 2wsinhw + 2coshw$

Then $2wsinhw + 2coshw = E(x) - \left(\frac{dw}{dx}\right)^2 \leq E(x)$

And $E(0) = \left(\frac{1}{\sqrt{2}}\right)^2 + 2 = \frac{5}{2}$

Thus $2wsinhw + 2coshw \leq E(x) \leq E(0) = \frac{5}{2}$ (as $\frac{d}{dx}E(x) \leq 0$)

$\Rightarrow coshw \leq \frac{5}{4} - wsinhw \leq \frac{5}{4}$ (for $x \geq 0$),

as w & $sinhw$ always have the same sign (unless both are zero),

as required.