STEP 2013, Paper 2, Q2 – Solution (3 pages; 7/5/20)

[Almost entirely 'show that' results.]

(i) **1st part**

Let $y = 1 - x$,

so that $\int_0^1 x^{n-1}(1-x)^n dx = \int_1^0 (1-y)^{n-1} y^n (-dy)$

$= \int_0^1 y^n (1-y)^{n-1} dy = \int_0^1 x^n (1-x)^{n-1} dx$, as required (A)

2nd part

$I_{n-1} = \int_0^1 x^{n-1}(1-x)^{n-1} dx$

And $\int_0^1 x^{n-1}(1-x)^n dx$

$= \int_0^1 x^{n-1}(1-x)^{n-1} dx - \int_0^1 x^n (1-x)^{n-1} dx$

$= I_{n-1} - \int_0^1 x^{n-1}(1-x)^n dx$, from (A)

Hence $2\int_0^1 x^{n-1}(1-x)^n dx = I_{n-1}$, as required (B)

($n \geq 1 \Rightarrow I_{n-1}$ is defined)

3rd part

By Parts, $I_n = \int_0^1 x^n (1-x)^n dx$

$= \left[x^n \left(-\frac{1}{n+1}\right)(1-x)^{n+1}\right]_0^1 - \int_0^1 n x^{n-1} \left(-\frac{1}{n+1}\right)(1-x)^{n+1} dx$

$= \frac{n}{n+1}\int_0^1 x^{n-1}(1-x)^{n+1} dx$, as required

4th part

Hence $I_n = \frac{n}{n+1}\{\int_0^1 x^{n-1}(1-x)^n dx - \int_0^1 x^n(1-x)^n dx\}$

$= \frac{n}{n+1}\left\{\frac{1}{2}I_{n-1} - I_n\right\}$, from (B)

$\Rightarrow I_n\{(n+1)+n\} = \frac{nI_{n-1}}{2}$

$\Rightarrow I_n = \frac{n}{2(2n+1)}I_{n-1}$, as required (C)

(ii) From (C), $I_n = \frac{n}{2(2n+1)} \cdot \frac{n-1}{2(2n-1)} \cdot \frac{n-2}{2(2n-3)} \cdots \frac{1}{2(3)} I_0$

$I_0 = \int_0^1 dx = 1$, so that $I_n = \frac{(n!)^2}{(n!)(2^n)(2n+1)(2n-1)\ldots 3}$

$= \frac{(n!)^2(2n)(2n-2)\ldots 2}{(n!)(2^n)(2n+1)!} = \frac{(n!)^2(2^n)(n!)}{(n!)(2^n)(2n+1)!} = \frac{(n!)^2}{(2n+1)!}$

(iii) **1st part**

$I_{\frac{1}{2}} = \int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} dx$

Let $x = sin^2\theta$, so that $dx = 2sin\theta cos\theta d\theta$

and $I_{\frac{1}{2}} = \int_0^{\frac{\pi}{2}} sin\theta cos\theta(2sin\theta cos\theta)d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} sin^2(2\theta)d\theta$

$= \frac{1}{4}\int_0^{\frac{\pi}{2}} 1 - cos4\theta \; d\theta$

$= \frac{1}{4}[\theta - \frac{1}{4}sin4\theta]_0^{\frac{\pi}{2}}$

$= \frac{1}{4}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{8}$

2nd part

As (C) doesn't require n to be an integer,

$$I_{\frac{3}{2}} = \frac{\left(\frac{3}{2}\right)}{2(4)} I_{\frac{1}{2}} = \frac{3}{16} \cdot \frac{\pi}{8} = \frac{3\pi}{128}$$