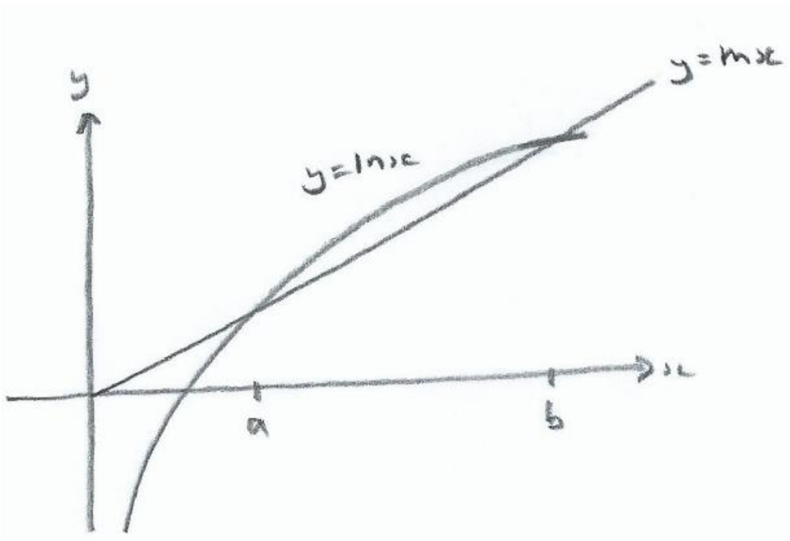
STEP 2013, Paper 2, Q1 – Sol'n (4 pages; 11/6/20)

(i) **1st part**

When $y = mx$ touches $y = lnx$, the gradient of $y = lnx$ equals m, so that $\frac{1}{x} = m$. Also, $mx = lnx$.

So $m\left(\frac{1}{m}\right) = \ln(\frac{1}{m})$, and hence $\frac{1}{m} = e$, and $m = \frac{1}{e}$

2nd part



As $mx = lnx$ at the points of intersection,

$m = \frac{lna}{a}$ & $m = \frac{lnb}{b}$, so that $blna = alnb$,

and hence $\ln(a^b) = \ln(b^a)$,

so that $a^b = b^a$, as required.

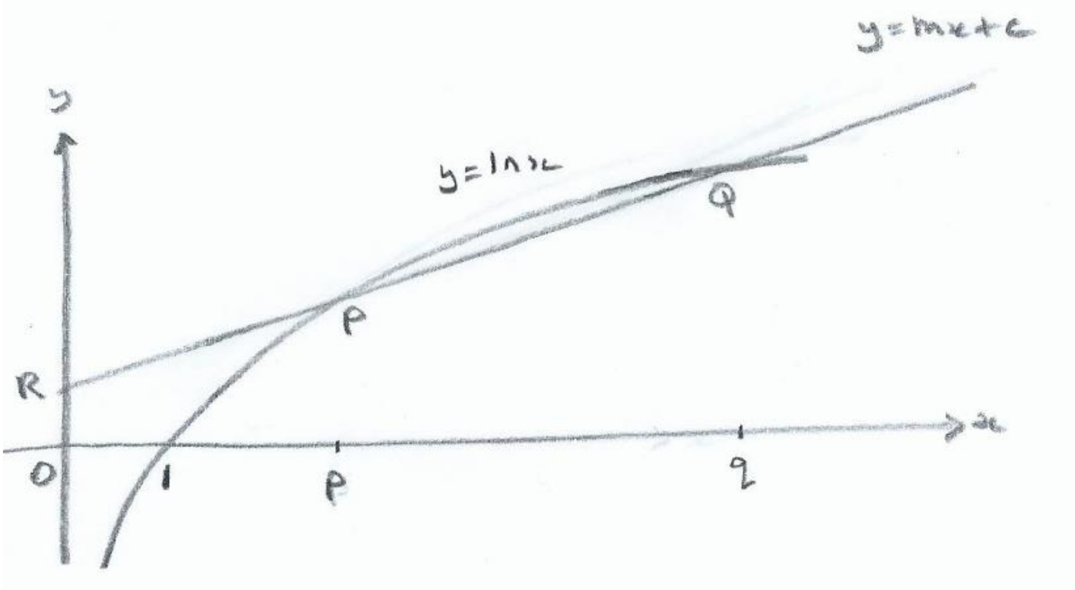
3rd part

At the point where $y = mx$ touches $y = lnx$, $x = \frac{1}{m}$

(from the working to the 1st part), so that $x = e$.

Thus from the previous diagram we see that $a < e < b$, as required.

(ii)

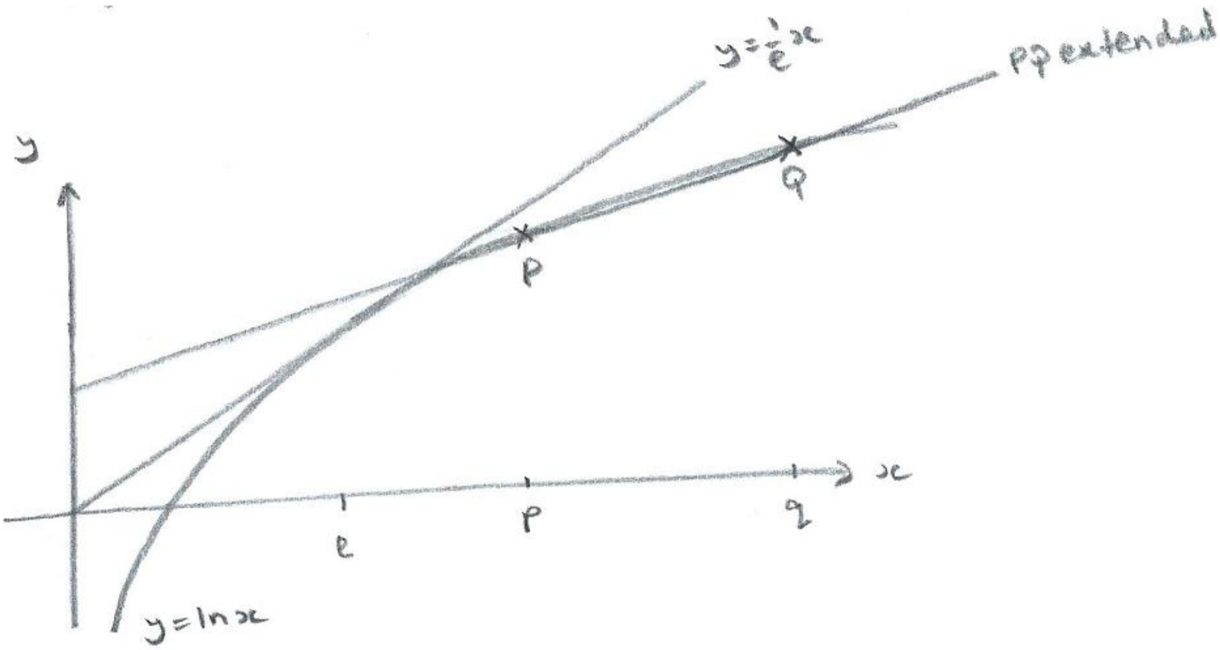


Referring to the diagram, in order for the line $y = mx + c$ (with $c > 0$) to intersect $y = lnx$ twice, m must be positive, and p must be greater than 1.

$p^q > q^p \Leftrightarrow qlnp > plnq \Leftrightarrow \frac{lnp}{p} > \frac{lnq}{q}$

As $\frac{lnp}{p}$ is the gradient of the line from the Origin to the point (p, lnp), and similarly for q, we can see from the diagram that $\frac{lnp}{p} > \frac{lnq}{q}$ (consider the triangle OPQ), and so it follows that $p^q > q^p$, as required.

(iii) **1st part**



From the 3rd part of (i), $y = \frac{1}{e}x$ touches $y = lnx$ when $x = e$.

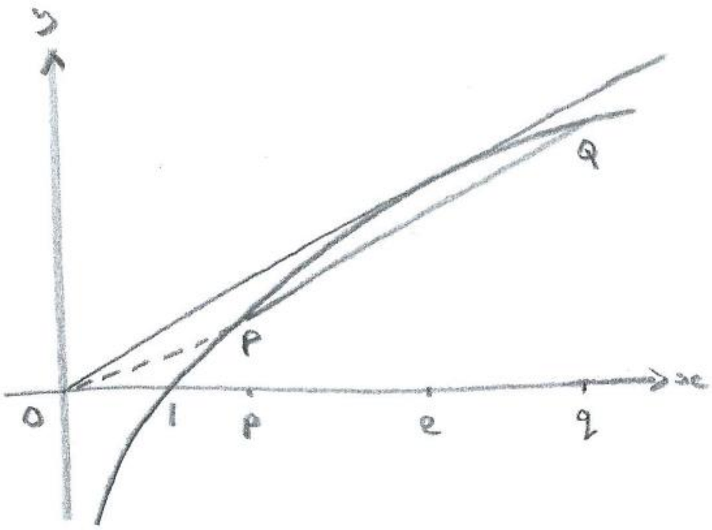
Then, we can see from the diagram that, as $p \geq e$, the tangent to $y = lnx$ at $x = p$ will be flatter than the tangent at $x = e$, and so the tangent at $x = p$ will cross the y-axis at a positive value of y. As the line though PQ is flatter still, it will also cross the y-axis at a positive value of y.

2nd part

Setting $p = e$ and $q = \pi$, (ii) $\Rightarrow e^{\pi} > \pi^{e}$.

(iv) From the earlier parts, the gradient of the tangent to $y = lnx$ at $x = e$ is $\frac{1}{e}$. And $\frac{lnq - lnp}{q-p} = \frac{1}{e} \Rightarrow PQ$ is parallel to this tangent.

And $q^p > p^q \Leftrightarrow plnq > qlnp \Leftrightarrow \frac{lnq}{q} > \frac{lnp}{p}$



First of all, note that $e \leq p < q$ isn't possible, as PQ will be flatter than the tangent at $x = e$.

And $p < q \leq e$ isn't possible, as PQ will be steeper than the tangent at $x = e$.

So the only possible scenarios are:

Case 1: $1 \leq p < e < q$ (as in the diagram above)

Case 2: $0 < p < 1 < e < q$

Case 1: $\boldsymbol{1 \leq p < e < q}$

From the diagram, considering the triangle OPQ, we can see that OQ is steeper than OP; ie $\frac{lnq}{q} > \frac{lnp}{p}$; therefore $q^p > p^q$.

Case 2: $\boldsymbol{0 < p < 1 < e < q}$

The same argument applies (in this case, OP has a negative gradient, and is therefore clearly less steep than OQ).