STEP 2013, Paper 1, Q4 – Solution (2 pages; 14/5/20)

(i) Let $u = tanx$, so that $du = sec^2x\ dx$, and

$$\int_0^{\frac{\pi}{4}} tan^n x\ sec^2 x\ dx = \int_0^1 u^n du = \left[\frac{1}{n+1}u^{n+1}\right]_0^1 = \frac{1}{n+1}$$, as required.

Let $u = secx$, so that $du = tanxsecx\ dx$, and

$$\int_0^{\frac{\pi}{4}} sec^n x\ tanx\ dx = \int_1^{\sqrt{2}} u^{n-1} du = \left[\frac{1}{n}u^n\right]_1^{\sqrt{2}} = \frac{1}{n}(\sqrt{2}^n - 1),$$

as required.

(ii) By Parts, $\int_0^{\frac{\pi}{4}} xsec^4x\ tanx\ dx = [x.\frac{1}{4}sec^4x]_0^{\frac{\pi}{4}}$

$- \int_0^{\frac{\pi}{4}} \frac{1}{4}sec^4x\ dx$, from the working to the 2nd integral in (i)

$= \frac{\pi}{16}\sqrt{2}^4 - \frac{1}{4}\int_0^{\frac{\pi}{4}} sec^2x(tan^2x + 1)dx$

$= \frac{\pi}{16}(4) - \frac{1}{4} \cdot \frac{1}{2+1} - \frac{1}{4}[tanx]_0^{\frac{\pi}{4}}$, from the 1st integral in (i)

$= \frac{\pi}{4} - \frac{1}{12} - \frac{1}{4}(1 - 0)$

$= \frac{\pi}{4} - \frac{1}{3}$

By Parts, $\int_0^{\frac{\pi}{4}} x^2sec^2xtanx\ dx = [x^2.\frac{1}{2}sec^2x]_0^{\frac{\pi}{4}}$

$- \int_0^{\frac{\pi}{4}} 2x.\frac{1}{2}sec^2x\ dx$, from the working to the 2nd integral in (i)

$$= \frac{\pi^2}{32}(2) - \int_0^{\frac{\pi}{4}} xsec^2x\,dx$$

$$= \frac{\pi^2}{32}(2) - \{[xtanx]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} tanx\,dx\}$$

$$= \frac{\pi^2}{16} - \frac{\pi}{4} + [\ln|secx|]_0^{\frac{\pi}{4}}]$$

$$= \frac{\pi^2}{16} - \frac{\pi}{4} + [\ln(\sqrt{2}) - \ln(1)]$$

$$= \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}ln2$$