

STEP 2013, Paper 1, Q13 – Solution (2 pages; 15/5/20)

Number of ways of selecting 7 integers is $\binom{52}{7}$

Number of ways of selecting 3 pairs and one other integer is

$\binom{26}{3}(46)$ (as there are 26 possible pairs adding up to 53, and 46 left over once 3 pairs have been chosen)

$$\begin{aligned} \text{So } P(X = 3) &= \frac{\binom{26}{3}(46)}{\binom{52}{7}} = \frac{26(25)(24)(46)(7)(6)(5)(4)(3)(2)}{3(2)(52)(51)(50)(49)(48)(47)(46)} \\ &= \frac{7(6)(5)(4)}{2(51)(2)(49)(2)(47)} = \frac{(3)(5)}{51(7)(47)} = \frac{5}{7(17)(47)} = \frac{5}{5593} \end{aligned}$$

Number of ways of selecting 2 pairs and 3 other integers (avoiding further pairs being created) is

$\binom{26}{2} \cdot \frac{(48)(46)(44)}{3!}$ (dividing by 3! to remove duplication, as order isn't important)

$$\begin{aligned} \text{So } P(X = 2) &= \frac{\binom{26}{2}(48)(46)(44)}{(6)\binom{52}{7}} = \frac{(26)(25)(48)(46)(44)(7)(6)(5)(4)(3)(2)}{2(6)(52)(51)(50)(49)(48)(47)(46)} \\ &= \frac{(44)(7)(5)(4)(3)}{(2)(51)(2)(49)(47)} = \frac{(44)(5)}{(17)(7)(47)} = \frac{220}{5593} \end{aligned}$$

Number of ways of selecting 1 pair and 5 other integers is

$(26) \cdot \frac{(50)(48)(46)(44)(42)}{5!}$

$$\text{So } P(X = 1) = \frac{26(50)(48)(46)(44)(42)}{(5!)\binom{52}{7}}$$

$$= \frac{(26)(50)(48)(46)(44)(42)(7!)}{(5!)(52)(51)(50)(49)(48)(47)(46)} = \frac{(44)(42)(7)(6)}{(2)(51)(49)(47)}$$

$$= \frac{(44)(42)}{(7)(17)(47)} = \frac{1680+168}{5593} = \frac{1848}{5593}$$

Number of ways of selecting 0 pairs is $\frac{(52)(50)(48)(46)(44)(42)(40)}{7!}$

$$\text{So } P(X = 0) = \frac{(52)(50)(48)(46)(44)(42)(40)}{(7!)\binom{52}{7}}$$

$$= \frac{(52)(50)(48)(46)(44)(42)(40)}{(52)(51)(50)(49)(48)(47)(46)} = \frac{(44)(42)(40)}{(51)(49)(47)} = \frac{(44)(2)(40)}{(17)(7)(47)} = \frac{88(40)}{5593}$$

$$= \frac{3520}{5593}$$

$$[\text{Check: } \frac{3520}{5593} + \frac{1848}{5593} + \frac{220}{5593} + \frac{5}{5593} = 1]$$

$$E(X) = (0) \frac{3520}{5593} + (1) \frac{1848}{5593} + (2) \frac{220}{5593} + (3) \frac{5}{5593}$$

$$= \frac{(1848+440+15)}{5593} = \frac{2303}{5593} = \frac{7(329)}{17(329)} = \frac{7}{17}, \text{ as required}$$