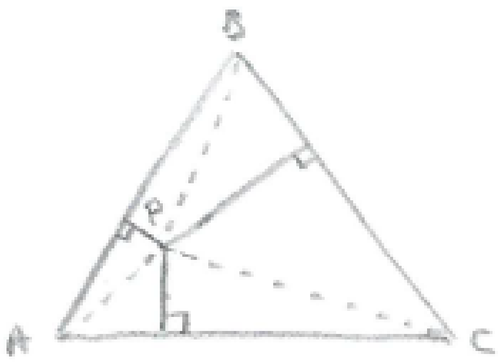
STEP 2012, Paper 3, Q12 – Solution (2 pages; 16/7/18)

(i)

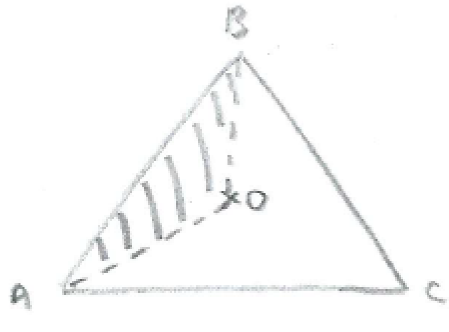


Area of equilateral triangle $= \frac{1}{2}(\sqrt{2})(1)$

Area also = Area(ABP)+Area(BCP)+Area(CAP)

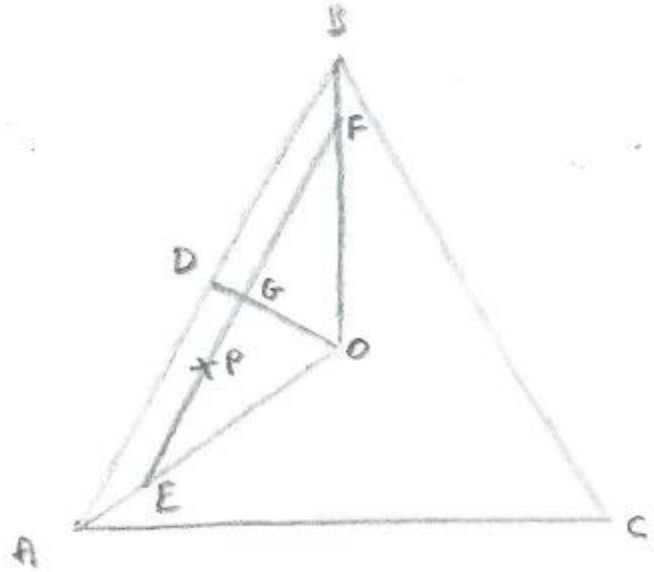
$= \frac{1}{2}(\sqrt{2})x_1 + \frac{1}{2}(\sqrt{2})x_2 + \frac{1}{2}(\sqrt{2})x_3$

Hence $1 = x_1 + x_2 + x_3$, as required.



In the diagram, O is the centre of mass of the triangle (where the angle bisectors meet).

To find $f(x)$: Without loss of generality, label the diagram so that P lies within the area ABO, as shown below, where $DG = x$.



To find the pdf, consider the cdf, $F(x) = P(X < x)$

$F(x) = P(\text{P lies in ABFE}|\text{P lies in ABO})$

$= 1 - P(\text{P lies in EFO}|\text{P lies in ABO})$

$= 1 - \frac{\text{area EFO}}{\text{area ABO}}$

EFO and ABO are similar triangles, so that $\frac{\text{area EFO}}{\text{area ABO}} = \left(\frac{OG}{OD}\right)^2$

$= \left(\frac{\frac{1}{3} - x}{1/3}\right)^2$

Then $F(x) = 1 - (1 - 3x)^2$

and $f(x) = \frac{d}{dx}(F(x)) = -2(1 - 3x)(-3) = 6(1 - 3x)$

for $0 \leq x \leq 1/3$ (as $\text{OD} = \frac{1}{3}$)

$E(X) = \int_0^{1/3} x.6(1 - 3x)dx = \int_0^{1/3} 6x - 18x^2 dx$

$= [3x^2 - 6x^3]_0^{1/3} = \frac{1}{3} - \frac{2}{9} = \frac{1}{9}$

(iii) We now want the equivalent result for similar volumes, so that $F(x) = 1 - \left(\frac{\frac{1}{4} - x}{\frac{1}{4}}\right)^3 = 1 - (1 - 4x)^3$

Then $g(x)$[following the notation of the official sol'ns] $=$

$-3(1-4x)^2(-4) = 12(1-4x)^2$

So $E(X) = \int_0^{1/4} x.12(1-4x)^2 dx = \int_0^{1/4} 12x - 96x^2 + 192x^3 dx$

$= [6x^2 - 32x^3 + 48x^4]_0^{1/4} = \frac{6}{16} - \frac{32}{64} + \frac{48}{256} = \frac{6-8+3}{16} = \frac{1}{16}$

[In the ER, 'cpf' should be 'cdf' (the 'cumulative distribution function').]